MATH 234 EXAM#2

Name Solutions

Work all 7 problems. Show ALL your work. No calculators may be used on this exam. Total : 50 points.

1. (10 points) Find the derivatives. Do not simplify your answer.

a) $\frac{\partial}{\partial x}(e^{-xy}\sin(x+yz)) = -y e^{-xy}\sin(x+yz) + e^{-xy}\cos(x+yz)$

b) $\frac{\partial^2}{\partial x \partial y}(e^x \ln xy)$

$$\frac{\partial}{\partial y}(e^x \ln xy) = e^x \frac{1}{xy} \cdot x = e^x \frac{1}{y}$$

$$\frac{\partial^2}{\partial x \partial y}(e^x \ln xy) = \frac{\partial}{\partial x}\left(e^x \frac{1}{y}\right) = e^x \cdot \frac{1}{y}$$

c) In what direction is the derivative of $f(x,y) = x^2y^2 + 2y^3$ at $P(2,1)$ equal to zero?

$$\nabla f = (2xy^2)\vec{i} + (2yx^2 + 6y^2)\vec{j}$$

$$\nabla f \Big|_{(2,1)} = 4\vec{i} + 14\vec{j}$$

$D_{\vec{u}} f \Big|_{(2,1)} = (4\vec{i} + 14\vec{j}) \cdot (u_1\vec{i} + u_2\vec{j}) = 0$ when

$u_1 = -14$, $u_2 = 4$ so $\vec{u}$ is any multiple of $(-14, 4)$

d) Find the direction of most rapid increase of $f(x,y,z) = \ln(x^3 + y^3 - 1) - y + 3z$ at $(1,1,4)$.

$$\nabla f = \left(\frac{1}{x^3+y^3-1} \cdot 3x^2\right)\vec{i} + \left(\frac{1}{x^3+y^3-1} \cdot 3y^2 - 1\right)\vec{j} + 3\vec{k}$$

$$\nabla f \Big|_{(1,1,4)} = 3\vec{i} + 2\vec{j} + 3\vec{k}$$

direction of most rapid increase is $(3, 2, 3)$

2. (4 points) By considering different paths of approach show that:

$$\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{2x^2+3y^2}{xy}$$

does not exist.

(1) Consider the line $x = y$

$$\lim_{(x,x)\to(0,0)} \frac{2x^2+3y^2}{xy} = \lim_{(x,x)\to(0,0)} \frac{2x^2+3x^2}{x^2} = 5$$

(2) Consider the line $x = -y$

$$\lim_{(x,-x)\to(0,0)} \frac{2x^2+3y^2}{xy} = \lim_{(x,-x)\to(0,0)} \frac{2x^2+3y^2}{-x^2} = -5$$

since $5 \neq -5$ the limit does not exist.

3. (4 points) Assuming that the equation:

$$1 - x - y^2 - \sin xy = 0,$$

defines y as a differentiable function of x, find $\frac{dy}{dx}$ at $P(0,1)$.

Diff. wrt x:

$$0 = -1 - 2y\frac{dy}{dx} - \cos(xy)\cdot\left(y + x\frac{dy}{dx}\right)$$

at $(0,1)$ $x=0$, $y=1$

$$0 = -1 - 2\left.\frac{dy}{dx}\right|_{(0,1)} - 1 + 0\cdot\left.\frac{dy}{dx}\right|_{(0,1)}$$

$$= -1 - 2\left.\frac{dy}{dx}\right|_{(0,1)} - 1$$

so $\left.\frac{dy}{dx}\right|_{(0,1)} = -1$

4. (6 points) Find the equation of the tangent plane to the surface

$$x^2 + y^2 - 2xy - x + 3y - z = -4$$

at the point $P\,(2, -3, 18)$.

Set $f(x, y, z) = x^2 + y^2 - 2xy - x + 3y - z$

then ∇f is normal to the surface

$\nabla f = (2x - 2y - 1)\vec{i} + (2y - 2x + 3)\vec{j} - \vec{k}$

$\nabla f \big|_{(2,-3,18)} = 9\vec{i} - 7\vec{j} - \vec{k}$

equation of tangent plane:

$9(x-2) - 7(y+3) - (z-18) = 0$

5. (6 points) Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$ if $w = x^3 + y^2 z + y$ and $\begin{cases} x = s^2 \\ y = s + t^2 \\ z = st \end{cases}$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial t}$

$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial s}$

where $\dfrac{\partial w}{\partial x} = 3x^2 = 3s^4$ $\quad \dfrac{\partial w}{\partial y} = 2yz + 1 = 2(s+t^2)st + 1$

$\dfrac{\partial w}{\partial z} = y^2 = (s+t^2)^2$

so $\dfrac{\partial w}{\partial t} = 3s^4 \cdot 0 + \left(2(s+t^2)st + 1\right) 2t + (s+t^2)^2 s$

$\dfrac{\partial w}{\partial s} = 3s^4 \cdot 2s + (2(s+t^2)st + 1)\,1 + (s+t^2)^2 t$

6. (10 points) Find all the local maximum, local minimum, and saddle points of

$$f(x, y) = 2x^3 + 3xy + 2y^3.$$

Since f is diff'ble everywhere critical points occur where $f_x = 0$ & $f_y = 0$

But $f_x = 6x^2 + 3y$ $\quad f_y = 3x + 6y^2$

so critical pts occur where

$$6x^2 + 3y = 0 \qquad 3x + 6y^2 = 0$$

or

$$2x^2 + y = 0 \qquad x + 2y^2 = 0$$

Thus,

$$2(-2y^2)^2 + y = 0$$

or

$$8y^4 + y = 0$$

Case (i) $y = 0$, $x = 0$ is a critical pt.

Case (ii) $y \neq 0$ so $8y^3 = -1$ $\quad y^3 = -1/8$ $\quad y = -1/2$

then $x = -1/2$

$(-1/2, -1/2)$ is a critical pt.

Compute $f_{xx} = 12x$, $f_{yy} = 12y$ $\quad f_{xy} = 3$

At $(0,0)$ $\quad f_{xx} f_{yy} - f_{xy}^2 = -9 < 0$

$(0,0)$ is a saddle point

At $(-1/2, -1/2)$ $\quad f_{xx} f_{yy} - (f_{xy})^2 = 36 - 9 > 0$

& $f_{xx} < 0$

$(-1/2, -1/2)$ is a local max.

7. (10 points) Find the points on the sphere $x^2 + y^2 + z^2 = 6$ where

$$f(x, y, z) = 2x - y + z$$

has its maximum and minimum values.

Method of lagrange multipliers: Find (x, y) where

(i) $\nabla f = \lambda \nabla g$ $\qquad g(x, y, z) = x^2 + y^2 + z^2 - 6$

(ii) $g(x, y, z) = 0$

$\nabla f = 2\vec{i} - \vec{j} + \vec{k} \qquad \nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

(i) $2\vec{i} - \vec{j} + \vec{k} = \lambda(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$

or $\quad 2 = \lambda 2x, \quad -1 = \lambda 2y \quad 1 = \lambda 2k$

so $\quad x = \frac{1}{\lambda} \qquad y = -\frac{1}{2\lambda} \qquad z = \frac{1}{2\lambda}$

(ii) $x^2 + y^2 + z^2 = 6$ becomes

$$\left(\frac{1}{\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 6$$

or

$$1 + \frac{1}{4} + \frac{1}{4} = 6\lambda^2$$

$$\frac{6}{4} = 6\lambda^2$$

$$\lambda = \pm\frac{1}{2}$$

Two cases: (a) $\lambda = \frac{1}{2}$ then $x = 2, \; y = -1, \; z = 1$

(b) $\lambda = -\frac{1}{2}$ then $x = -2, \; y = 1, \; z = -1$

Finally when $(2, -1, 1)$ $\quad f(2, -1, 1) = 4 + 1 + 1 = 6$

$(-2, 1, -1)$ $\quad f(-2, 1, -1) = -4 - 1 - 1 = -6$

$\therefore$ $(2, -1, 1)$ max pt.

$(-2, 1, -1)$ min pt.