Math 234, Practice Test #4

Show your work in all the problems.

1. Evaluate the line integral
$$\int_C 2xy\,dx + (x^2 + y^2)\,dy$$
where C is the circular arc given by
$$\mathbf{r}(t) = (x(t), y(t)) = (\cos t, \sin t)\ ,\ 0 \le t \le \frac{\pi}{2}$$

2. Find a potential function for the vector field $\mathbf{F} = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y + \pi)\mathbf{j}$

3. Use Green's theorem to evaluate the integral
$$\oint_C 3y\,dx + 2x\,dy$$
where C is the boundary of the region $0 \le x \le \pi$, $0 \le y \le \sin x$

4. Let S be the surface consisting of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$, $z \ge 0$ and the circle $x^2 + y^2 \le a$ in the xy-plane, let $\mathbf{n}$ be the outward unit normal vector, and let $\mathbf{F}$ be the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. Use the divergence theorem to compute
$$\int\int_S \mathbf{F} \bullet \mathbf{n}\,d\sigma.$$

5. Find the area of the portion of the paraboloid $x = 4 - y^2 - z^2$ that lies above the ring $1 \le y^2 + z^2 \le 5$ in the yz-plane.

6. *(Extra credit problem)* Find the work done by the force
$$\mathbf{F}(x, y) = (ye^{xy}, xe^{xy})$$
as it acts on a particle moving from $P = (-1, 0)$ to $Q = (1, 0)$ along the semicircular arc C given by $\mathbf{r}(t) = (-\cos t, \sin t)$, $0 \le t \le \pi$.

7. *(Extra credit problem)* Use the surface integral in Stokes' theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction
$$\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$$
C is the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \ge 0$, counterclockwise when viewed from above.

Solutions

1. We compute

$$\begin{aligned}
\int_C 2xy\,dx + (x^2+y^2)\,dy &= \int_0^{\pi/2} (2\cos t \sin t)\left(\frac{d}{dt}\cos t\right)\,dt + \\
&\quad + \int_0^{\pi/2} (\cos^2 t + \sin^2 t)\left(\frac{d}{dt}\sin t\right)\,dt \\
&= -2\int_0^{\pi/2} \sin^2 t \cos t\,dt + \int_0^{\pi/2} \cos t\,dt \\
&= -\frac{2}{3}\sin^3 t\Big|_0^{\pi/2} + \sin t|_0^{\pi/2} \\
&= -\frac{2}{3} + 1 \\
&= \frac{1}{3}
\end{aligned}$$

2. The potential function ϕ has to satisfy

$$\frac{\partial\phi}{\partial x} = \cos y + y\,\cos x \quad \text{and} \quad \frac{\partial\phi}{\partial y} = \sin x - x\,\sin y + \pi$$

The first equation implies that

$$\phi(x,y) = x\,\cos y + y\,\sin x + f(y)$$

where f is a function depending on y only. In order to find it differentiate with respect to y and use the second equation

$$\frac{\partial\phi}{\partial y} = -x\,\sin y + \sin x + f'(y) = \sin x - x\,\sin y + \pi.$$

We see that $f'(y) = \pi$, i.e. the function f equals $f(y) = \pi y + c$ where c is constant. Hence

$$\phi(x,y) = x\,\cos y + y\,\sin x + \pi y + c\,, \quad c \text{ is a constant}$$

3. Green's theorem asserts that

$$\oint_C M dx + N dy = \int\int_R (N_x - M_y)\,dx\,dy$$

Hence $M = 3y$, $N = 2x$ and $N_x = 2$, $M_y = 3$ so that the given integral equals

$$-\int_0^{\pi}\int_0^{\sin x} dy\, dx = -\int_0^{\pi} \sin x\, dx = \cos\pi - \cos 0 = -2$$

4. The surface S encloses a domain which we denote by D. We compute

$$\nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2+y^2+z^2).$$

We use the divergence theorem, and we calculate the triple integral using spherical coordinates

$$\begin{aligned}
\int\int_S \mathbf{F}\bullet\mathbf{n}\, d\sigma &= \int\int\int_D \nabla\bullet\mathbf{F}\, dV \\
&= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} (3\rho^2)\rho^2 d\rho\, d\phi\, d\theta \\
&= 3\int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^5}{5}\sin\phi\Big|_0^a\, d\phi\, d\theta \\
&= \frac{3a^5}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\, d\phi\, d\theta \\
&= \frac{3a^5}{5}\int_0^{2\pi} d\theta \\
&= \frac{6\pi a^5}{5}
\end{aligned}$$

5. Let $f(x, y, z) = x + y^2 + z^2$ so that the paraboloid is given by the equation $f(x, y, z) = 4$. Then the surface area is givern by

$$\int\int_R \frac{|\nabla f|}{|\nabla f \bullet \mathbf{p}|} dy\, dz$$

where R is the region in the yz-plane given by $1 \le y^2 + z^2 \le 5$, and where $\mathbf{p}$ is a vector of length one perpendicular to the region R, for example $\mathbf{p} = (1, 0, 0)$ would do the job. We also compute

$$\nabla f = (1, 2y, 2z)\,,\ |\nabla f| = \sqrt{1+4y^2+4z^2}\,,\ |\nabla f \bullet \mathbf{p}| = 1$$

It will be convenient to use polar coordinates

$$y = r\,\cos\theta \quad , \quad z = r\sin\theta$$

for the calculation of the double integral since the region R is then simply decribed by $1 \le r^2 \le 5$. Then

$$\begin{aligned}
\int\int_R \frac{|\nabla f|}{|\nabla f \bullet \mathbf{p}|} dy\, dz &= \int_0^{2\pi} \int_1^{\sqrt{5}} \sqrt{1+4r^2}\, r\, dr\, d\theta \\
&\quad \text{substitute } u = 1+4r^2,\ du = 8r\, dr \\
&= \frac{1}{8} \int_0^{2\pi} \int_5^{21} \sqrt{u}\, du\, d\theta \\
&= \frac{\pi}{4} \frac{2}{3} u^{3/2} \Big|_5^{21} \\
&= \frac{\pi}{6}(21\sqrt{21} - 5\sqrt{5})
\end{aligned}$$

6. The objective is to compute the line integral

$$\int_C \mathbf{F} \bullet d\mathbf{r}$$

but it is a bad idea to compute it directly (try it to see why). There is an easier way. The key observation is that the vector field $\mathbf{F} = (M, N) = (ye^{xy}, xe^{xy})$ is *conservative* since $N_x = M_y = xye^{xy} + e^{xy}$. Because the field is conservative the line integral does not depend on the curve C, i.e. if D is another curve connecting P and Q then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_D \mathbf{F} \bullet d\mathbf{r}$$

so we may choose an easier path from P to Q than the circular arc. A good one is the straight line segment

$$D: \ \mathbf{r}(t) = (x(t), y(t)) = (t, 0)\ , -1 \le t \le 1$$

on the x-axis. We get

$$d\mathbf{r} = (1, 0)\, dt\ ,\ \mathbf{F}(\mathbf{r}(t)) = (0, t)$$

and

$$\int_D \mathbf{F} \bullet \mathbf{r} = \int_{-1}^{1} 0\, dt = 0$$

Another way to compute the integral is to find a potential function ϕ. Then

$$\int_C \mathbf{F} \bullet \mathbf{r} = \phi(Q) - \phi(P)$$

The potential function ϕ must satisfy

$$\phi_x = ye^{xy} \quad \text{and} \quad \phi_y = xe^{xy}$$

so that $\phi(x,y) = e^{xy}$ and

$$\int_C \mathbf{F} \bullet \mathbf{r} = \phi(Q) - \phi(P) = \phi(1,0) - \phi(-1,0) = 1 - 1 = 0.$$

7. Stokes theorem asserts that

$$\int_C \mathbf{F} \bullet d\mathbf{r} \;=\; \int\int_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, d\sigma$$

where S is a two-sided surface with boundary C and unit normal vector $\mathbf{n}$. Note that we are free to choose S as we like (as long as it has C as its boundary). The easiest choice would be the horizontal disk with radius 2 (draw a picture of the situation, hard to do on the computer)

$$S : x^2 + y^2 \leq 4\ ,\ \ z = \sqrt{12}.$$

In order to traverse C counterclockwise when viewed from above we need to choose $\mathbf{n} = (0,0,1)$. We compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = -\frac{\partial}{\partial y}(x^2y^3)\mathbf{k} = -3y^2x^2\mathbf{k}$$

The surface S is given by the equation $f(x,y,z) = z = \sqrt{12}$, and its 'shadow region' R in the xy-plane is the circle $x^2 + y^2 \leq 4$. We choose $\mathbf{p} = (0,0,1)$ (perpendicular to R in the xy-plane) so that

$$\frac{\nabla f}{|\nabla f|} = (0,0,1)$$

and with polar coordinates in the xy-plane

$$\begin{aligned}
\int\int_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, d\sigma &= \int\int_R (\nabla \times \mathbf{F}) \bullet \frac{\nabla f}{|\nabla f \bullet \mathbf{p}|}\, dx\, dy \\
&= \int\int_R -3y^2x^2\mathbf{k} \bullet \mathbf{k}\, dx\, dy \\
&= -3 \int_0^{2\pi} \int_0^2 r^4 \cos^2\theta \sin^2\theta\, r\, dr\, d\theta
\end{aligned}$$

$$
\begin{aligned}
&= -3\int_0^{2\pi} (\cos\theta\sin\theta)^2 \left[\frac{r^6}{6}\right]\Bigg|_0^2 \, d\theta \\
&= -32\int_0^{2\pi} (\cos\theta\sin\theta)^2 \, d\theta \\
&= -8\int_0^{2\pi} (2\cos\theta\sin\theta)^2 \, d\theta \\
&= -8\int_0^{2\pi} (\sin(2\theta))^2 \, d\theta \\
&\quad \text{substitute } u = 2\theta,\ du = 2\,d\theta \\
&= -4\int_0^{4\pi} \sin^2 u \, du \\
&\quad \text{trig identity } \sin^2 u = \frac{1}{2}(1-\cos(2u)) \\
&= -4\left[\frac{u}{2} - \frac{\sin(2u)}{4}\right]\Bigg|_0^{4\pi} \\
&= -8\pi
\end{aligned}
$$