MTH 234
Michigan State University
Department of Mathematics

Name: ____________________ **PID:** ______________ **Section No:** ______

Problem	**Total**	**Score**
1	16	
2	16	
3	17	
4	17	
5	16	
6	17	
7	17	
8	17	
9	17	
10	16	
11	17	
12	17	
Total	200	

Michigan State University
Department of Mathematics

Name: ____________________ **PID:** ______________ **Section No:** ______

Signature: ____________________

Total Score:

1. Check that you have pages 1 through 16 and that none are blank.
2. Fill in the information at the top of the page.
3. You will need a pen or pencil and this booklet for the exam. Please clear everything else from your desk.
4. The use of calculators, cell phones, or any other electronic device as an aid to writing this exam is strictly prohibited.
5. The grading of this exam is based on your method. **Show all of your work.** (There are problems however that will be graded right or wrong.) If you need additional space, use the backs of the exam pages.
6. If you present different answers, the worst answer will be graded.
7. **Box your answers.**

1. (16 points)

(a) Find a unit vector in the opposite direction of $\boldsymbol{v} = \langle 1, 2, 3 \rangle$.

(b) Find the scalar projection of $\boldsymbol{w} = \langle 1, -1, 2 \rangle$ onto $\boldsymbol{v}$.

(c) Find the vector projection of $\boldsymbol{w}$ onto $\boldsymbol{v}$.

SOLUTION:

(a)

$$\boldsymbol{u} = -\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = -\frac{1}{\sqrt{1+4+9}} \langle 1, 2, 3 \rangle \quad \Rightarrow \quad \boxed{\boldsymbol{u} = -\frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle}.$$

(b)

$$P_v(w) = \frac{\boldsymbol{w} \cdot \boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(1 - 2 + 6)}{\sqrt{1+1+4}} \quad \Rightarrow \quad \boxed{P_v(w) = \frac{5}{\sqrt{6}}}.$$

(c)

$$\boldsymbol{P}_v(w) = P_v(w) \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{5}{\sqrt{6}} \frac{1}{\sqrt{6}} \langle 1, -1, 2 \rangle \quad \Rightarrow \quad \boxed{\boldsymbol{P}_v(w) = \frac{5}{6} \langle 1, -1, 2 \rangle}.$$

2. (16 points) Find the equation of the plane that contains the lines $\boldsymbol{r}_1(t) = \langle 1, 2, 3\rangle t$ and $\boldsymbol{r}_2(t) = \langle 1, 1, 0\rangle + \langle 1, 2, 3\rangle t$.

SOLUTION: $P_0 = (1, 1, 0)$ is in the plane. $P_1 = (1, 2, 3) = \boldsymbol{r}_1(t = 1)$ is also in the plane.

Therefore, $\overrightarrow{P_0P_1} = \langle 0, 1, 3\rangle$ is tangent to the plane.

$\boldsymbol{v} = \langle 1, 2, 3\rangle$ is also tangent to the plane. Then, the normal vector to the plane $\boldsymbol{n}$ can be computed as follows:

$$\boldsymbol{n} = \boldsymbol{v} \times \overrightarrow{P_0P_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{vmatrix} = (6-3)\boldsymbol{i} - (3-0)\boldsymbol{j} + (1-0)\boldsymbol{k} = \langle 3, -3, 1\rangle.$$

Then, the equation of the plane can be constructed with $P_0 = (1, 1, 0)$ and $\boldsymbol{n} = \langle 3, -3, 1\rangle$ as follows:

$$\boxed{3(x-1) - 3(y-1) + z = 0} \quad \Leftrightarrow \quad \boxed{3x - 3y + z = 0}.$$

◁

3. (17 points) A particle moves along the curve $\boldsymbol{r}(t) = \langle \sin(2t^2), t^3, \cos(2t^2) \rangle$, for $t \geq 0$.

(a) Find the velocity $\boldsymbol{v}(t)$ and acceleration $\boldsymbol{a}(t)$ functions of the particle.

(b) Find the arc length function for the curve $\boldsymbol{r}(t)$ measured from the point where $t = 0$, in the direction of increasing t.

SOLUTION:

(a)

$$\begin{aligned}
\boldsymbol{v}(t) &= \langle 4t\cos(2t^2), 3t^2, -4t\sin(2t^2) \rangle, \\
\boldsymbol{a}(t) &= \langle \big[4\cos(2t^2) - 4t(4t)\sin(2t^2)\big], 6t, \big[-4\sin(2t) - (4t)(4t)\cos(2t^2)\big] \rangle \\
\boldsymbol{a}(t) &= \langle \big[4\cos(2t^2) - 16t^2\sin(2t^2)\big], 6t, -\big[4\sin(2t) + 16t^2\cos(2t^2)\big] \rangle.
\end{aligned}$$

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(b)

$$\begin{aligned}
s(t) &= \int_0^t |\boldsymbol{v}(\tau)|\, d\tau, \\
&= \int_0^t \sqrt{16\tau^2\cos^2(2\tau^2) + 9\tau^4 + 16\tau^2\sin^2(2\tau^2)}\, d\tau \\
&= \int_0^t \sqrt{16\tau^2 + 9\tau^4}\, d\tau \\
&= \int_0^t \sqrt{16 + 9\tau^2}\, \tau d\tau, \quad u = 16 + 9\tau^2, \quad du = 18\tau\, d\tau \\
&= \frac{1}{18}\int_{16}^{16+9t^2} u^{1/2}\, du \\
&= \frac{1}{18}\,\frac{2}{3}u^{3/2}\Big|_{16}^{16+9t^2} \\
&= \frac{1}{27}\big[(16 + 9t^2)^{3/2} - (16)^{3/2}\big] \\
&= \frac{1}{27}\big[(16 + 9t^2)^{3/2} - 4^3\big]
\end{aligned}$$

We then conclude that

$$\boxed{s(t) = \frac{1}{27}\big[(16 + 9t^2)^{3/2} - 4^3\big]}.$$

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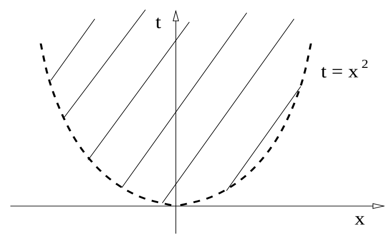
4. (17 points)

(a) Find and sketch the domain of the function $f(x,t) = \ln(t - x^2)$.

(b) Determine whether the function f above is solution of the wave equation $f_{tt} - f_{xx} = 0$.

SOLUTION:

(a) The domain of f is given in the picture.



(b) We must compute $\partial_t f$ and $\partial_{xx} f$;

$$\partial_t f = \frac{1}{t - x^2} \quad \Rightarrow \quad \partial_{tt} f = -\frac{1}{(t - x^2)^2};$$

$$\partial_x f = \frac{-2x}{t - x^2} \quad \Rightarrow \quad \partial_{xx} f = \frac{-2}{t - x^2} + \frac{2x}{(t - x^2)^2}\,(-2x)$$
$$= \frac{-2}{t - x^2} - \frac{4x^2}{(t - x^2)^2}.$$

Therefore,

$$\partial_{tt} f - \partial_{xx} f = -\frac{1}{(t - x^2)^2} + \frac{2}{t - x^2} + \frac{4x^2}{(t - x^2)^2}$$
$$= \frac{2}{t - x^2} + \frac{(-1 + 4x^2)}{(t - x^2)^2}$$
$$\neq 0.$$

We conclude that f is not solution of the wave equation above .

5. (16 points)

(a) Find the tangent plane approximation of $f(x, y) = \sin(2x + 5y)$ at the point $(-5, 2)$.

(b) Use the linear approximation computed above to approximate the value of $f(-4.8, 2.1)$.

SOLUTION:

(a) The linear approximation of function f near $(-5, 2)$ is

$$L_{(-5,2)}(x, y) = (\partial_x f)_{(-5,2)}\,(x + 5) + (\partial_y f)_{(-5,2)}\,(y - 2) + f(-5, 2).$$

We need to compute three numbers: $f(-5, 2)$, $(\partial_x f)_{(-5,2)}$, and $(\partial_y f)_{(-5,2)}$. Since

$$\partial_x f = 2\cos(2x + 5y) \quad \Rightarrow \quad (\partial_x f)_{(-5,2)} = 2.$$

$$\partial_y f = 5\cos(2x + 5y) \quad \Rightarrow \quad (\partial_y f)_{(-5,2)} = 5.$$

Finally, $f(-5, 2) = 0$, so we conclude

$$\boxed{L_{(-5,2)} = 2(x + 5) + 5(y - 2)}.$$

(b) We use the approximation $f(-4.8, 2.1) \simeq L_{(-5,1)}(-4.8, 2.1)$, that is

$$f(-4.8, 2.1) \simeq 2(0.2) + 5(0.1) = 0.4 + 0.5 \quad \Rightarrow \quad \boxed{f(-4.8, 2.1) \simeq 0.9}.$$

6. (17 points) Find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + 3y^2 - 2xy$ in the triangle formed by the lines $y = 0$, $x = 1$ and $y = x$.

SOLUTION: we first compute the local extrema of function f.

$$\nabla f = \langle (2x - 2y), (6y - 2x) \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad \begin{cases} y = x \\ 3y = x \end{cases} \quad \Rightarrow \quad P_0 = (0, 0), \quad f(0, 0) = 0.$$

We now need to study f on the boundary of the triangle. We add to the candidate list the vertex points. Since $(0, 0)$ is already in the list we only need:

$$P_1 = (1, 0), \quad f(1, 0) = 1, \qquad P_2 = (1, 1), \quad f(1, 1) = 2.$$

We now study f on the boundary. On the line $y = 0$, $x \in [0, 1]$ we have

$$g(x) = f(x, 0) = x^2 \quad \Rightarrow \quad 0 = g'(x) = 2x \quad \Rightarrow \quad x = 0,$$

so we reobtain the point $P_0 = (0, 0)$. On the line $y =\in [0, 1]$, $x = 1$ we have

$$g(y) = f(1, y) = 3y^2 - 2y + 1 \quad \Rightarrow \quad 0 = g'(y) = 6y - 2 \quad \Rightarrow \quad y = 1/3,$$

so we obtain the point $P_4 = (1, 1/3)$, and $f(1, 1/3) = 2/3$. Finally we study f on the line $y = x$, $x \in [0, 1]$,

$$g(x) = f(x, x) = x^2 + 3x^2 - 2x^2 = 2x^2 \quad \Rightarrow \quad 0 = g'(x) = 4x \quad \Rightarrow \quad x = 0,$$

so we reobtain $(0, 0)$. We therefore conclude:

$$\boxed{\text{Absolute minimum: } P_0 = (0, 0)}, \qquad \boxed{\text{Absolute maximum: } P_2 = (1, 1)}.$$

◁

7. (17 points)

(a) Sketch the region of integration, D, whose area is given by the double integral

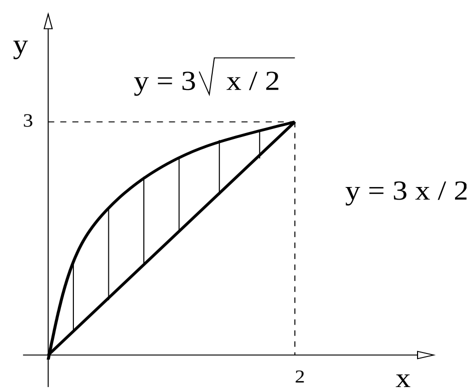
$$\int\int_D dA = \int_0^2 \int_{\frac{3}{2}x}^{3\sqrt{x/2}} dy\, dx.$$

(b) Compute the double integral given in (a).

(c) Change the order of integration in the integral given in (a).

SOLUTION:

(a)



(b)

$$\begin{aligned}\int\int_D dA &= \int_0^2 \int_{\frac{3}{2}x}^{3\sqrt{x/2}} dy\, dx,\\ &= \int_0^2 \Big[\frac{3}{\sqrt{2}}x^{1/2} - \frac{3}{2}x\Big]\, dx,\\ &= \frac{3}{\sqrt{2}}\frac{2}{3}\Big(x^{3/2}\Big|_0^2\Big) - \frac{3}{4}\Big(x^2\Big|_0^2\Big),\\ &= \sqrt{2}\big(\sqrt{2}\big)^3 - \frac{3}{4}4,\\ &= 4 - 3,\\ &= 1.\end{aligned}$$

(c)

$$\int\int_D dA = \int_0^3 \int_{\frac{2}{9}y^2}^{\frac{2}{3}y} dx\, dy.$$

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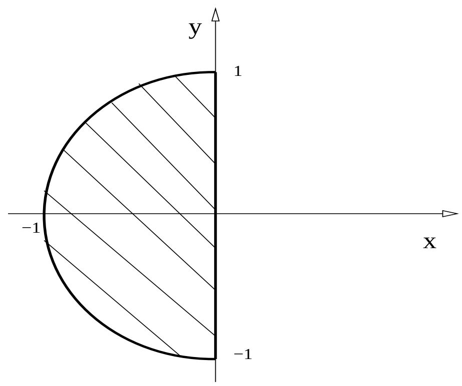
8. (17 points) Transform to polar coordinates and then evaluate the integral

$$I = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{0} \ln(x^2 + y^2 + 1)\, dx\, dy.$$

SOLUTION: The integration region is:



Therefore,

$$\begin{aligned} I &= \int_{\pi/2}^{3\pi/2} \int_0^1 \ln(1 + r^2)\, r\, dr\, d\theta \\ &= \pi \int_0^1 \ln(1 + r^2)\, r\, dr\, d\theta, \quad u = 1 + r^2, \quad du = 2r\, dr; \\ &= \frac{\pi}{2} \int_1^2 \ln(u)\, du \\ &= \frac{\pi}{2} \big(u \ln(u) - u\big)\Big|_1^2 \\ &= \frac{\pi}{2} \big[(2\ln(2) - 2) - (0 - 1)\big] \\ &= \frac{\pi}{2} \big[2\ln(2) - 1\big]. \end{aligned}$$

We conclude that $\boxed{I = \pi \left(\ln(2) - \frac{1}{2}\right)}$. ◁

9. (17 points) Find the component $\overline{z}$ of the centroid for a wire lying along the the curve given by $\boldsymbol{r}(t) = \langle t\cos(t), t\sin(t), (2\sqrt{2}/3)t^{3/2}\rangle$, for $t \in [0,1]$.

Solution: We need to compute

$$\overline{z} = \frac{1}{M}\int_0^1 z\,|\boldsymbol{r}'(t)|\,dt, \qquad M = \int_0^1 |\boldsymbol{r}'(t)|\,dt.$$

We start with $\boldsymbol{r}'(t)$:

$$\boldsymbol{r}'(t) = \big\langle \big[\cos(t) - t\sin(t)\big], \big[\sin(t) + t\cos(t)\big], \sqrt{2}\,t^{1/2}\big\rangle.$$

Therefore,

$$\begin{aligned} |\boldsymbol{r}'(t)|^2 &= \cos^2(t) + t^2\sin^2(t) - 2t\sin(t)\cos(t) \\ &\quad + \sin^2(t) + t^2\cos^2(t) + 2t\sin(t)\cos(t) + 2t \\ &= 1 + t^2 + 2t \\ &= (1+t)^2 \quad \Rightarrow \quad |\boldsymbol{r}'(t)| = 1 + t. \end{aligned}$$

Now we can compute M as follows,

$$M = \int_0^1 (1+t)\,dt = \Big(t + \frac{t^2}{2}\Big)\Big|_0^1 = 1 + \frac{1}{2} \quad \Rightarrow \quad M = \frac{3}{2}.$$

Hence, $\overline{z}$ is given by

$$\overline{z} = \frac{2}{3}\int_0^1 \frac{2}{3}\sqrt{2}t^{3/2}\,(1+t)\,dt = \sqrt{2}\int_0^1 \big(t^{3/2} + t^{5/2}\big)\,dt = \sqrt{2}\Big(\frac{2}{5}\,t^{5/2} + \frac{2}{7}\,t^{7/2}\Big)\Big|_0^1.$$

That is,

$$\overline{z} = \sqrt{2}\Big(\frac{2}{5} + \frac{2}{7}\Big) = \sqrt{2}\frac{(14+10)}{35} \quad \Rightarrow \quad \boxed{\overline{z} = \sqrt{2}\,\frac{24}{35}}.$$

10. (16 points) Use the Green Theorem area formula to find the area of the region enclosed by the curve $\boldsymbol{r}(t) = \langle \cos^2(t), \sin^2(t) \rangle$ for $t \in [0, \pi/2]$. (16.4.23).

Solution: The tangential form of the Green Theorem, $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S \big(\partial_x F_y - \partial_y F_x\big)\, dA$, applied to $\boldsymbol{F}(x, y) = \langle 0, x \rangle$ implies the formula

$$A(S) = \iint_S dA = \oint_C x\, dy.$$

The curve given in this problem defines a surface S, and the area of this surface is

$$A(S) = \int_0^{\pi/2} x(t)\, y'(t)\, dt = \int_0^{\pi/2} \cos^2(t)\, \big[2\sin(t)\cos(t)\big]\, dt$$

$$A(S) = 2\int_0^{\pi/2} \cos^3(t)\, \sin(t)\, dt = -2\int_0^{\pi/2} \frac{1}{4}\frac{d}{dt}\cos^4(t)\, dt = -\frac{1}{2}\cos^4(t)\Big|_0^{\pi/2} = -\frac{1}{2}(0 - 1).$$

We conclude that $\boxed{A(S) = \frac{1}{2}}$. ◁

11. (17 points) Find the flux of $\nabla \times \boldsymbol{F}$ outward through the surface S, where $\boldsymbol{F} = \langle -y, x, x^2 \rangle$ and $S = \{x^2 + y^2 = a^2,\ z \in [0, h]\} \cup \{x^2 + y^2 \leqslant a^2,\ z = h\}$.

SOLUTION: We use the Stokes Theorem: $\displaystyle\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

The surface S is the cylinder walls and its cover at $z = h$. Therefore, the curve C is the circle $x^2 + y^2 = a^2$ at $z = 0$.

That circle can be parametrized (counterclockwise) as $\boldsymbol{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ for $t \in [0, 2\pi]$.

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(t) \cdot \boldsymbol{r}'(t)\, dt,$$

where $\boldsymbol{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t)\rangle$ and $\boldsymbol{r}'(t) = \langle -a\sin(t), a\cos(t), 0\rangle$. Hence

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma = \int_0^{2\pi} \boldsymbol{F}(t) \cdot \boldsymbol{r}'(t)\, dt,$$

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma = \int_0^{2\pi} \big(a^2\sin^2(t) + a^2\cos^2(t)\big)\, dt = \int_0^{2\pi} a^2\, dt.$$

We conclude that $\displaystyle\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma = 2\pi a^2$. ◁

12. (17 points) Find the outward flux of the field $\boldsymbol{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

SOLUTION: We use the Divergence Theorem: $\displaystyle\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \iiint_D (\nabla \cdot \boldsymbol{F}) dv.$

$$\nabla \cdot \boldsymbol{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2x - 2x + 3x \quad \Rightarrow \quad \nabla \cdot \boldsymbol{F} = 3x.$$

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \iiint_D (\nabla \cdot \boldsymbol{F}) dv = \iint_D 3x\, dx\, dy\, dz.$$

It is convenient to use spherical coordinates:

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \big[3\rho \sin(\phi) \cos(\phi)\big]\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \Big[\int_0^{\pi/2} \cos(\theta)\, d\theta\Big] \Big[\int_0^{\pi/2} \sin^2(\phi)\, d\phi\Big] \Big[\int_0^2 3\rho^3\, d\rho\Big]$$

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \Big[\sin(\theta)\Big|_0^{\pi/2}\Big] \Big[\frac{1}{2} \int_0^{\pi/2} \big(1 - \cos(2\phi)\big)\, d\phi\Big] \Big[\frac{3}{4}\rho^4\Big|_0^2\Big]$$

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = (1)\frac{1}{2}\Big(\frac{\pi}{2}\Big)(12) \quad \Rightarrow \quad \boxed{\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = 3\pi}.$$

◁