Name: ______________________________ **ID Number:** ____________________

TA: ______________________________ **Section Time:** ____________________

MTH 234
Exam 4: Practice
December 7, 2010
50 minutes
Sects: 16.1-16.5,
16.7, 16.8.

No calculators or any other devices allowed.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. Box your answers.

1. (20 points) Find the potential function for $\boldsymbol{F} = \left\langle \frac{2x}{y}, \frac{(1-x^2)}{y^2} \right\rangle$, for $y > 0$.

SOLUTION: We must find a scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ solution of the equations

$$\partial_x f = \frac{2x}{y}, \qquad \partial_y f = \frac{1-x^2}{y^2}.$$

From the first equation we obtain $f(x,y) = \frac{x^2}{y} + g(y)$. Introduce this expression for f into the second equation above,

$$-\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} = \frac{1}{y^2} - \frac{x^2}{y^2} \quad \Rightarrow \quad g'(y) = \frac{1}{y^2}.$$

We conclude that $g(y) = -\frac{1}{y} + c$, where c is an arbitrary constant. Therefore,

$$f(x,y) = \frac{x^2}{y} - \frac{1}{y} + c \quad \Rightarrow \quad \boxed{f(x,y) = \frac{x^2-1}{y} + c}.$$

2. (20 points) Use the Green Theorem in the plane to show that line integral given by $\oint_C \big[xy^2\,dx + (x^2y + 2x)\,dy\big]$ around any square depends only on the area of the square and not on its location in the plane.

SOLUTION:

The Green Theorem in the plane says that

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy$$

We use this Theorem for the field $\boldsymbol{F}$ such that

$$\oint_C \big[xy^2\,dx + (x^2y + 2x)\,dy\big] = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r} \quad\Rightarrow\quad \boldsymbol{F} = \langle xy^2, (x^2y + 2x)\rangle.$$

Therefore,

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy = \iint_S \big[(2xy + 2) - 2xy\big]\,dx\,dy = 2\iint_S dx\,dy.$$

Since $\iint_S dx\,dy = A(S)$ is the area of the integration region, the line integral satisfies the equation:

$$\boxed{\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 2A(S)}\,.$$

Therefore, the line integral is independent of the position of the integration region in space, it depends only on the area $A(S)$ of the integration region.

3. (20 points) Write an integral which gives the surface area of the surface cut from the hemisphere $x^2 + y^2 + z^2 = 6$, with $z \geqslant 0$ by the cylinder $(x-1)^2 + y^2 = 1$. Your final answer should be written in cylindrical coordinates. Do not evaluate the integral.

SOLUTION: We must compute the integral $A(S) = \iint_S d\sigma$, where

$$S = \{x^2 + y^2 + z^2 = 6, \quad (x-1)^2 + y^2 \leqslant 1\}.$$

The area of this surface in space is given by

$$A(S) = \iint_S d\sigma = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \boldsymbol{k}|}\, dx\, dy,$$

where $R = \{(x-1)^2 + y^2 \leqslant 1,\ z = 0\}$ is a disk on the $z = 0$ plane centered at $(1, 0)$ with radius $a = 1$, and the function $f : \mathbb{R}^3 \to \mathbb{R}$ is any function whose level surface $f = 0$ defines S. We consider the simplest f given by

$$f(x, y, z) = x^2 + y^2 + Z^2 - 6 \quad \Rightarrow \quad \nabla f = 2\langle x, y, z\rangle.$$

Therefore,

$$|\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{6}, \qquad |\nabla f \cdot \boldsymbol{k}| = 2z \quad \Rightarrow \quad d\sigma = \frac{\sqrt{6}}{z}\, dx\, dy$$

where $z = \sqrt{6 - x^2 - y^2}$. We then obtain,

$$A(S) = \iint_R \frac{\sqrt{6}}{\sqrt{6 - x^2 - y^2}}\, dx\, dy.$$

We now need to express this integral in cylindrical coordinates (r, θ, z). The border of the region R in Cartesian coordinates is given by

$$(x-1)^2 + y^2 = 1 \quad \Leftrightarrow \quad x^2 - 2x + 1 + y^2 = 1 \quad \Leftrightarrow \quad x^2 + y^2 = 2x,$$

which in cylindrical coordinates is given by

$$r^2 = 2r\cos(\theta) \quad \Leftrightarrow \quad r = 2\cos(\theta).$$

Therefore, we conclude that

$$\boxed{A(S) = 2\int_0^{\pi/2}\int_0^{2\cos(\theta)} \frac{\sqrt{6}}{\sqrt{6 - r^2}}\, r\, dr\, d\theta},$$

where the factor 2 in front of the integral comes from the fact that we are integrating on only half the region R.

4. (20 points) Use the Stokes Theorem to compute the line integral of the vector field $\boldsymbol{F} = \langle x^2 y, 1, z\rangle$ along the path C given by the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, with $z \geqslant 0$, counterclockwise when viewed from above.

SOLUTION: Stokes' Theorem says that $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma$. In our case we have the path $C = \{x^2 + y^2 + z^2 = 16,\ \text{and } x^2 + y^2 = 4,\ z \geqslant 0\}$. This curve can be also given by

$$C = \{x^2 + y^2 = 4, \quad z = \sqrt{12}\}.$$

In Stokes Theorem we are free to choose any surface S in space whose boundary is C. We choose the simplest one, the flat disk

$$S = \{x^2 + y^2 \leqslant 4, \quad z = \sqrt{12}\}.$$

(The surface integral in Stokes' Theorem is simple since the surface S is flat.) We need to compute $\nabla \times \boldsymbol{F}$, that is,

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 y & 1 & z \end{vmatrix} = \langle 0, 0, -x^2\rangle.$$

Therefore,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{k}\, dx\, dy = \iint_S (-x^2)\, dx\, dy.$$

We use cylindrical coordinates to compute the integral above:

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{2\pi} \int_0^2 -r^3 \cos^2(\theta)\, r\, dr\, d\theta \\ &= -\Big[\int_0^{2\pi} \cos^2(\theta)\, d\theta\Big]\Big[\int_0^2 r^3\, dr\Big] \\ &= -\Big[\int_0^{2\pi} \frac{1}{2}(1 + \cos(2\theta))\, d\theta\Big]\Big[\frac{r^4}{4}\Big|_0^2\Big] \\ &= -4\pi. \end{aligned}$$

We conclude that $\boxed{\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = -4\pi}$.

5. (20 points) Use the Divergence Theorem to find the outward flux of the field $\boldsymbol{F} = \sqrt{x^2+y^2+z^2}\,\langle x, y, z\rangle$ across the boundary of the region $D = \{1 \leqslant x^2+y^2+z^2 \leqslant 2\}$.

SOLUTION: The Divergence Theorem says that $\displaystyle\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,d\sigma = \iiint_D (\nabla\cdot\boldsymbol{f})\,dv$. We need to compute $nabla\cdot\boldsymbol{F}$, that is,

$$F_x = x\sqrt{x^2+y^2+z^2} \quad\Rightarrow\quad \partial_x F_x = \sqrt{x^2+y^2+z^2} + x\frac{1}{2}\frac{1}{\sqrt{x^2+y^2+z^2}}2x$$

that is,

$$\partial_x F_x = \sqrt{x^2+y^2+z^2} + \frac{x^2}{\sqrt{x^2+y^2+z^2}}.$$

Analogously,

$$\partial_y F_y = \sqrt{x^2+y^2+z^2} + \frac{y^2}{\sqrt{x^2+y^2+z^2}}, \qquad \partial_z F_z = \sqrt{x^2+y^2+z^2} + \frac{z^2}{\sqrt{x^2+y^2+z^2}}.$$

Therefore,

$$\nabla\cdot\boldsymbol{F} = 3\sqrt{x^2+y^2+z^2} + \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}. \quad\Rightarrow\quad \nabla\cdot\boldsymbol{F} = 4\sqrt{x^2+y^2+z^2}.$$

Since the region D has spherical symmetry, we use spherical coordinates (ρ, ϕ, θ) to compute the triple integral,

$$\begin{aligned}
\iiint_D (\nabla\cdot\boldsymbol{F})\,dv &= \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}} (4\rho)\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta \\
&= 2\pi\Big[\int_0^{\pi}\sin(\phi)\,d\phi\Big]\Big[\int_1^{\sqrt{2}} 4\rho^3\,d\rho\Big] \\
&= 2\pi\Big(-\cos(\phi)\Big|_0^{\pi}\Big)\Big(\rho^4\Big|_1^{\sqrt{2}}\Big) \\
&= 2\pi(2)(4-1) \quad\Rightarrow\quad \boxed{\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,d\sigma = 12\pi}.
\end{aligned}$$

#	Pts	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Σ	100	