Name: ______________________________ **ID Number:** ____________________

TA: ______________________________ **Section Time:** ____________________

MTH 234
Exam 3: Practice
November 9, 2010
50 minutes
Sects: 15.1-15.4, 15.6

No calculators or any other devices allowed.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. ***Box your answers.***

1. (26 points) Consider the integral $\iint_D f(x,y)\,dA = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx.$

(a) (8 points) Sketch the region of integration.

(b) (8 points) Switch the order of integration in the above integral.

(c) (10 points) Compute the integral $\iint_D f(x,y)\,dA$ for the case $f(x,y) = xy$.

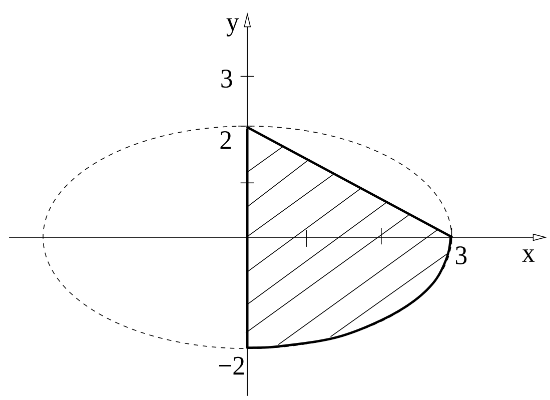
SOLUTION:

(a)

- The limits in x: $x \in [0,3]$.
- The limits in y:

  Upper limit: $y = 2\Big(1 - \dfrac{x}{3}\Big)$.

  Lower limit: $y = -2\sqrt{1 - \dfrac{x^2}{3^2}}$; so a part of the ellipse $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$.



(b) If we integrate first in x, we need to split the integral at $y = 0$. In the interval $y \in [-2, 0]$, the lower limit in x is $0 \leqslant x$. The upper limit comes from $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$, that is, $x = +3\sqrt{1 - \dfrac{y^2}{2^2}}$.

In the interval $y \in [0, 2]$, the lower limit in x is again $0 \leqslant x$. The upper limit comes from $y = 2\Big(1 - \dfrac{x}{3}\Big)$, that is, $x = 3\Big(1 - \dfrac{y}{2}\Big)$.

We then conclude:

$$\iint_D f(x,y)\,dA = \int_{-2}^{0} \int_0^{3\sqrt{1-\frac{y^2}{2^2}}} f(x,y)\,dx\,dy + \int_0^2 \int_0^{3(1-\frac{y}{2})} f(x,y)\,dx\,dy.$$

(c) This is a straightforward, albeit long, calculation. We can use any of the two order of integration we have for I. We choose the shorter one:

$$I = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} xy\, dy\, dx = \int_0^3 x \left(\frac{y^2}{2} \Big|_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} \right) dx,$$

$$I = \frac{1}{2} \int_0^3 x \left[4\left(1 - \frac{x}{3}\right)^2 - 4\left(1 - \frac{x^2}{3^2}\right) \right] dx = 2 \int_0^3 x \left(1 + \frac{x^2}{3^2} - 2\frac{x}{3} - 1 + \frac{x^2}{3^2} \right) dx,$$

$$I = 2 \int_0^3 x \left(2\frac{x^2}{3^2} - 2\frac{x}{3} \right) dx = \frac{4}{3^2} \int_0^3 \left(x^3 - 3x^2 \right) dx,$$

$$I = \frac{4}{3^2} \left(\frac{x^4}{4} - x^3 \right) \Big|_0^3 = \frac{4}{3^2} \left(\frac{3^4}{4} - 3^3 \right) = 4 \left(\frac{3^2}{4} - 3 \right),$$

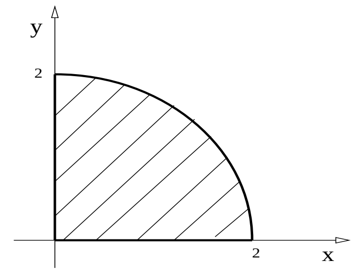
$$I = (9 - 12) = -5, \quad \Rightarrow \quad \boxed{I = -5}.$$

2. (20 points) Find the component x of the centroid vector in Cartesian coordinates in the plane of the region $R = \{(x, y) \in \mathbb{R}^2 \ : \ x \geqslant 0, \quad y \geqslant 0, \quad x^2 + y^2 \leqslant 2^2\}$.

SOLUTION:

If A denotes the area of the region, then the centroid vector $\overline{\boldsymbol{r}} = \langle \overline{x}, \overline{y} \rangle$ is given by:

$$\overline{x} = \frac{1}{A} \iint_R x\,dx\,dy, \qquad \overline{y} = \frac{1}{A} \iint_R y\,dx\,dy.$$



From the figure we see that the region is a quarter of a disk, hence $A = \frac{1}{4}\pi\, 2^2$, that is, $\boxed{A = \pi}$. Then, $\overline{x}$ is given by:

One way is:

$$\overline{x} = \frac{1}{A} \int_0^2 \int_0^{\sqrt{4-x^2}} x\,dy\,dx,$$
$$= \frac{1}{\pi} \int_0^2 \sqrt{4 - x^2}\, x\,dx,$$

substitution: $u = 4 - x^2, \ du = -2x\,dx$,

$$\overline{x} = \frac{1}{\pi} \int_4^0 -\frac{u^{1/2}}{2}\,du,$$
$$= \frac{1}{2\pi} \int_0^4 u^{1/2}\,du,$$
$$= \frac{1}{2\pi} \frac{2}{3} \Big(u^{3/2} \Big|_0^4 \Big),$$
$$= \frac{1}{3\pi} 8 \quad \Rightarrow \quad \boxed{\overline{x} = \frac{8}{3\pi}}.$$

Another ways is:

$$\overline{x} = \frac{1}{A} \int_0^2 \int_0^{\sqrt{4-y^2}} x\,dx\,dy,$$
$$= \frac{1}{\pi} \int_0^2 \Big(\frac{x^2}{2} \Big|_0^{\sqrt{4-y^2}} \Big)\,dy,$$
$$= \frac{1}{2\pi} \int_0^2 (4 - y^2)\,dy,$$
$$= \frac{1}{2\pi} \Big(4y \Big|_0^2 \Big) - \Big(\frac{y^3}{3} \Big|_0^2 \Big),$$
$$= \frac{1}{2\pi} \Big(8 - \frac{8}{3} \Big)$$
$$= \frac{4}{\pi} \frac{2}{3} \quad \Rightarrow \quad \boxed{\overline{x} = \frac{8}{3\pi}}.$$

Anyway it is correct.

3. (16 points) Transform to polar coordinates and then evaluate the integral

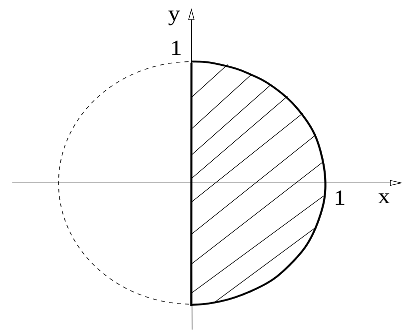
$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \left(x^2 + y^2\right)^{3/2} dx\, dy.$$

SOLUTION:

It is helpful to sketch the integration region:

- Limits in y: $y \in [-1, 1]$.
- Limits in x:

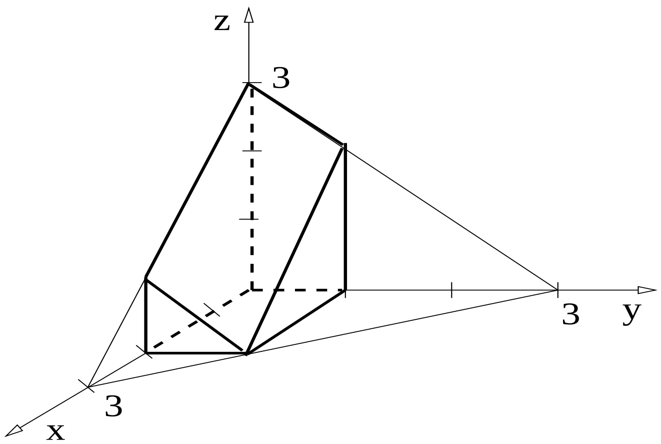
  Lower limit $x = 0$, upper limit the curve $x = \sqrt{1 - y^2}$, that is, the circle $x^2 + y^2 = 1$.



Therefore, the integral I in polar coordinates is the following

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \int_{0}^{1} \left(r^2\right)^{3/2} (r\, dr)\, d\theta, \\ &= \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_{0}^{1} r^4\, dr\right), \\ &= \pi \left(\frac{r^5}{5}\bigg|_0^1\right), \\ &= \frac{\pi}{5} \quad \Rightarrow \quad \boxed{I = \frac{\pi}{5}}. \end{aligned}$$

4. (18 points) Find the volume of a parallelepiped whose base is a rectangle in the $z = 0$ plane given by $0 \leqslant y \leqslant 1$ and $0 \leqslant x \leqslant 2$, while the top side lies in the plane $x+y+z = 3$.



SOLUTION:

$$
\begin{aligned}
V &= \int_0^2 \int_0^1 \int_0^{3-x-y} dz\, dy\, dx \\
&= \int_0^2 \int_0^1 (3 - x - y)\, dy\, dx, \\
&= \int_0^2 \left[(3 - x)\left(y\Big|_0^1 \right) - \frac{1}{2}\left(y^2\Big|_0^1 \right) \right] dx, \\
&= \int_0^2 \left(3 - x - \frac{1}{2} \right) dx, \\
&= \int_0^2 \left(\frac{5}{2} - x \right) dx, \\
&= \left[\frac{5}{2}\left(x\Big|_0^2 \right) - \frac{1}{2}\left(x^2\Big|_0^2 \right) \right], \\
&= 5 - 2, \\
&= 3 \quad \Rightarrow \quad \boxed{V = 3}.
\end{aligned}
$$

5. (20 points) Consider the region of $R \subset \mathbb{R}^3$ given by

$$R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant 1, \quad 0 \leqslant z \leqslant 1 + x^2 + y^2\}.$$

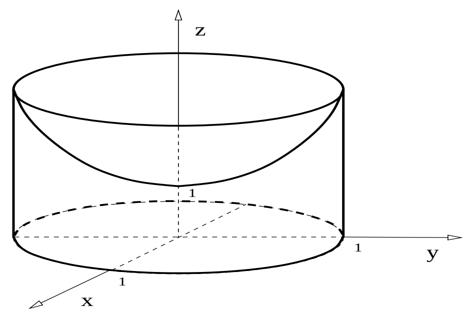
(a) (5 points) Sketch the region R.

(b) (15 points) Use cylindrical coordinates to compute the volume of that region.

SOLUTION:

(a)

- The condition $x^2 + y^2 \leqslant 1$ implies $r \leqslant 1$.
- The last condition is $0 \leqslant z \leqslant 1 + r^2$.
- No further conditions, so $\theta \in [0, 2\pi]$.



(b) The calculation is simple, once we have the appropriate integration limits.

$$V = \int_0^{2\pi} \int_0^1 \int_0^{1+r^2} dz\, r dr\, d\theta,$$

$$= 2\pi \int_0^1 (1 + r^2) r dr,$$

Substitution: $u = 1 + r^2, \quad du = 2r\, dr,$

$$V = 2\pi \int_1^2 \frac{u}{2} du,$$

$$= 2\pi \frac{1}{2} \Big(\frac{u^2}{2} \Big|_1^2 \Big),$$

$$= \pi \left(2 - \frac{1}{2}\right) \quad \Rightarrow \quad \boxed{V = \frac{3\pi}{2}}.$$

#	Pts	Score
1	26	
2	20	
3	16	
4	18	
5	20	
Σ	100	