Name: ______________________________ **ID Number:** ____________________

TA: ______________________________ **Section Time:** ____________________

MTH 234
Exam 1: Practice
September 21, 2010
50 minutes
Sects: 12.1-12.6.

No calculators or any other devices allowed.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. ***Box your answers.***

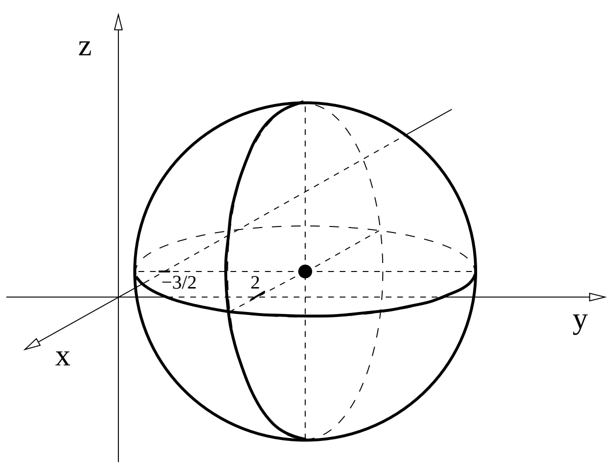
1. (20 points) Find the center and the radius of the sphere $x^2 + y^2 + z^2 + 3x - 4y = 0$. Sketch a qualitative picture of the sphere in a Cartesian coordinate system in $\mathbb{R}^3$.

SOLUTION: We complete squares to find the center point and the radius of the sphere.

$$\begin{aligned} 0 &= x^2 + y^2 + z^2 + 3x - 4y \\ &= \left[x^2 + 2\Big(\frac{3}{2}\Big)x + \frac{9}{4}\right] - \frac{9}{4} + \left[y^2 - 2(2y) + 4\right] - 4 + z^2 \\ &= \Big(x + \frac{3}{2}\Big)^2 + (y-2)^2 + z^2 - \frac{(9+16)}{4} \quad \Rightarrow \quad \boxed{\Big(x + \frac{3}{2}\Big)^2 + (y-2)^2 + z^2 = \Big(\frac{5}{2}\Big)^2}. \end{aligned}$$

We conclude that the sphere center point and radius are, respectively,

$$\boxed{P_0 = \Big(-\frac{3}{2}, 2, 0\Big)}, \qquad \boxed{r = \frac{5}{2}}.$$



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2. (10 points) Find the components in a 2-dimensional Cartesian coordinate system of a force vector with magnitude $|\boldsymbol{F}| = 3$ and having an angle $\theta = \pi/3$ with the positive horizontal axis.

Solution: If the vector $\boldsymbol{F}$ has an angle θ with the positive horizontal axis, then its components in a Cartesian coordinate system $\boldsymbol{F} = \langle F_x, F_y \rangle$ can be written as follows,

$$F_x = |\boldsymbol{F}|\,\cos(\theta), \qquad F_y = |\boldsymbol{F}|\,\sin(\theta).$$

Therefore

$$F_x = 3\cos(\pi/3) = \frac{3}{2}, \qquad F_y = 3\sin(\pi/3) = \frac{3\sqrt{3}}{2} \quad \Rightarrow \quad \boxed{\boldsymbol{F} = \frac{3}{2}\,\langle 1, \sqrt{3} \rangle}.$$

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3. (a) (5 points) Find a unit vector in the direction of $\boldsymbol{v} = \langle -1, 2, 1 \rangle$.

(b) (5 points) Find the scalar projection of $\boldsymbol{w} = \langle 1, 2, 1 \rangle$ onto $\boldsymbol{v}$.

(c) (5 points) Find the vector projection of $\boldsymbol{w}$ onto $\boldsymbol{v}$.

Solution:

(a)

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sqrt{1+4+1}}\,\langle -1, 2, 1 \rangle \quad \Rightarrow \quad \boxed{\boldsymbol{u} = \frac{1}{\sqrt{6}}\,\langle -1, 2, 1 \rangle}.$$

(b)

$$P_v(w) = \frac{\boldsymbol{w}\cdot\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-1+4+1)}{\sqrt{6}} \quad \Rightarrow \quad \boxed{P_v(w) = \frac{4}{\sqrt{6}}}.$$

(c)

$$\boldsymbol{P}_v(w) = P_v(w)\,\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{4}{\sqrt{6}}\,\frac{1}{\sqrt{6}}\,\langle -1, 2, 1 \rangle \quad \Rightarrow \quad \boxed{\boldsymbol{P}_v(w) = \frac{2}{3}\,\langle -1, 2, 1 \rangle}.$$

4. (a) (10 points) Find the intersection of the lines

$$\begin{aligned} x &= t, & x &= 2s+2, \\ y &= -t+2, & y &= s+3, \\ z &= t+1, & z &= 5s+6. \end{aligned}$$

(b) (10 points) Find the equation of the plane determined by these lines.

SOLUTION:

(a) The intersection point of these lines is solution of the equations:

$$t = 2s+2, \qquad -t+2 = s+3, \qquad t+1 = 5s+6,$$

Substitute t from the first equation into the second one:

$$-(2s+2)+2 = s+3 \quad \Rightarrow \quad -2s = s+3 \quad \Rightarrow \quad \boxed{s=-1} \quad \Rightarrow \quad \boxed{t=0}.$$

These lines intersect because these values of $t=0$ and $s=-1$ satisfy the third equation above: $0+1 = 5(-1)+6$. Therefore, the intersection point is $\boxed{P_0 = (0,2,1)}$.

(b) The lines intersect, so they determine a plane. The equation of the plane is fixed by a point in the plane and the normal vector. A point in the plane is the intersection point $P_0 = (0,2,1)$. (Any point on either line is ok, but we have already computed the intersection, so we use that point.) The normal vector $\boldsymbol{n}$ to the plane is the cross product of the line tangent vectors. We first rewrite the parametric equations for the lines into the vector equations:

$$\boldsymbol{r}(t) = \langle 0,2,1\rangle + \langle 1,-1,1\rangle\, t, \qquad \hat{\boldsymbol{r}}(s) = \langle 2,3,6\rangle + \langle 2,1,5\rangle\, s.$$

Therefore, a normal vector $\boldsymbol{n}$ to the plane is

$$\boldsymbol{n} = \boldsymbol{v}\times\boldsymbol{w}, \qquad \boldsymbol{v} = \langle 1,-1,1\rangle, \qquad \boldsymbol{w} = \langle 2,1,5\rangle,$$

that is,

$$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = \langle (-5-1), -(5-2), (1+2)\rangle \quad \Rightarrow \quad \boldsymbol{n} = \langle -6,-3,3\rangle.$$

A simpler solution is $\boxed{\hat{\boldsymbol{n}} = \langle -2,-1,1\rangle}$. We conclude that the equation of the plane is

$$-2(x-0) - (y-2) + (z-1) = 0 \quad \Rightarrow \quad \boxed{-2x - y + z = -1}.$$

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5. (a) (10 points) Find the cosine of the angle between the planes $2x - 3y + z = 1$ and $-x - 3y + 2z = 5$.

(b) (10 points) Find the vector equation of the line of intersection of the two planes given in (a).

SOLUTION:

(a) The angle between the plane is the angle between their normal vectors. These vectors are: $\boldsymbol{n} = \langle 2, -3, 1 \rangle$ and $\boldsymbol{N} = \langle -1, -3, 2 \rangle$. The cosine of the angle is:

$$\cos(\theta) = \frac{\boldsymbol{n} \cdot \boldsymbol{N}}{|\boldsymbol{n}|\,|\boldsymbol{N}|} = \frac{-2+9+2}{\sqrt{4+9+1}\,\sqrt{1+9+4}} \quad \Rightarrow \quad \boxed{\cos(\theta) = \frac{9}{14}}.$$

(b) The vector tangent to the line is $\boldsymbol{v} = \boldsymbol{n} \times \boldsymbol{N}$, that is,

$$\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -3 & 1 \\ -1 & -3 & 2 \end{vmatrix} = \langle (-6+3), -(4+1), (-6-3) \rangle = \langle -3, -5, -9 \rangle$$

A simpler tangent vector is $\hat{\boldsymbol{v}} = \langle 3, 5, 9 \rangle$. We now need to find a point in the intersection of the two planes. We first substitute the z coordinate from one plane into the other one:

$$z = 1 - 2x + 3y \quad \Rightarrow \quad -x - 3y + 2(1 - 2x + 3y) = 5 \quad \Rightarrow \quad -5x + 3y = 3.$$

We find a simple solution $x = 0$, then $y = 1$, and then $z = 4$. The intersection point is $P_1 = (0, 1, 4)$. The equation of the line is

$$\boxed{\boldsymbol{r}(t) = \langle 0, 1, 4 \rangle + \langle 3, 5, 9 \rangle\, t}.$$

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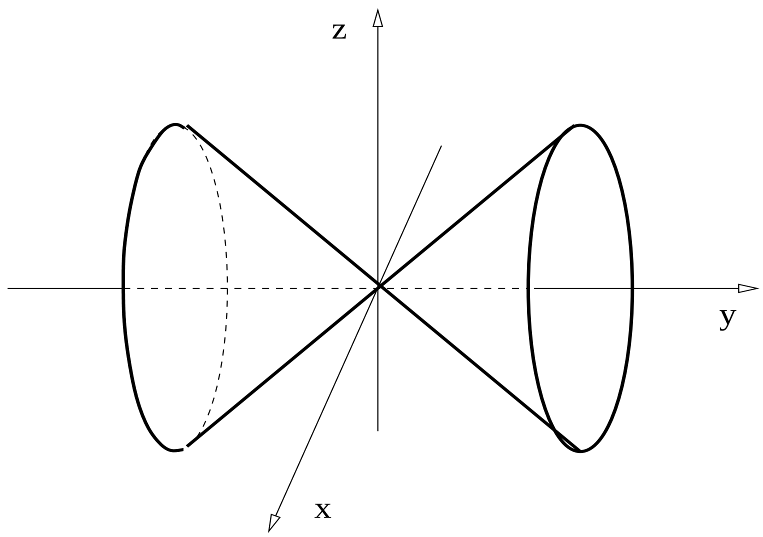
6. (15 points) Sketch a graph of the surface $x^2 - y^2 + \dfrac{z^2}{4} = 0$.

Solution: We can write the equation above as

$$y^2 = x^2 + \frac{z^2}{4} \quad \Rightarrow \quad y = \pm\sqrt{x^2 + \frac{z^2}{4}}.$$

Therefore, the surface is an elliptical cone along the y axis.



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