Chapter 1. Review problems

1. Linear systems

1. Let s be a real number, and consider the system

$$\begin{aligned} sx_1 - 2sx_2 &= -1, \\ 3x_1 + 6sx_2 &= 3. \end{aligned}$$

(a) Determine the values of the parameter s for which the system above has a unique solution.
(b) For all the values of s such that the system above has a unique solution, find that solution.

2. Find the values of k such that the system below has no solution; has one solution; has infinitely many solutions;

$$\begin{aligned} kx_1 + x_2 &= 1 \\ x_1 + kx_2 &= 1. \end{aligned}$$

3. Find a condition on the components of vector $\boldsymbol{b}$ such that the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -6 \\ 3 & 1 & -7 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

4. Use Cramer's rule to find the solution to the linear system

$$\begin{aligned} x_1 + 4x_2 - x_3 &= 1 \\ x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 3x_3 &= 0 \end{aligned}$$

2. Matrix algebra

1. Consider the matrix $A = \begin{bmatrix} 5 & 3 & s \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$ Find the value(s) of the constant s such that the matrix A is invertible. For such value(s) of s compute $\det(A^{-1})$.

2. Consider the matrix $A = \begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$. Find the coefficients $(A^{-1})_{12}$ and $(A^{-1})_{31}$ of the matrix A^{-1}, that is, of the inverse matrix of A. Show your work.

3. Which of the following matrices below is equal to $(A+B)^2$ for every square matrices A and B?

$$(B+A)^2, \quad A^2 + 2AB + B^2, \quad (A+B)(B+A), \quad A^2 + AB + BA + B^2, \quad A(A+B) + (A+B)B.$$

4. (a) Find the volume of the parallelepiped formed by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(b) The matrix $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ determines the linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$. Is this linear transformation one-to-one? Is it onto?

5. Find a matrix A solution of the matrix equation

$$AB + 2I = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}, \quad \text{with} \quad B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

6. (a) Find the values of the constant k such that $\det(A) = 0$, where,

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix}.$$

(b) Determine the values of k such that the following system of equations below has more than one solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + 3x_2 + kx_3 &= 0 \\ x_1 + kx_2 + 3x_3 &= 0 \end{aligned}$$

(c) Fix $k = 1$ and compute the coefficients $(A^{-1})_{1,2}$ and $(A^{-1})_{2,3}$ of A^{-1}. You do not need to compute the rest of the inverse matrix.

7. (a) Find the LU-factorization of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 7 \\ 6 & 9 & 12 \end{bmatrix}.$$

(b) Use the LU-factorization above to find the solutions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ of the linear systems $A\boldsymbol{x} = \boldsymbol{e}_1$ and $A\boldsymbol{x}_2 = \boldsymbol{e}_2$, where

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(c) Use the LU-factorization above to find A^1.

8. Explain in terms of volumes why $\det(3A) = 3^n \det(A)$ for any $n \times n$ matrix A.

3. Vector spaces

1. Find the dimension and give a basis of the vector space V on $\mathbb{R}^3$ given by

$$V = \left\{ \begin{bmatrix} -a + b + c - 3d \\ b + 3c - d \\ a + 2b + 8c \end{bmatrix} \quad \text{with} \quad a, b, c, d \in \mathbb{R} \right\}.$$

2. Determine whether the subset $V \subset \mathbb{R}^2$ is a subspace, where

$$V = \left\{ \begin{bmatrix} -a + 1 \\ a - 1 \end{bmatrix} \quad \text{with} \quad a \in \mathbb{R} \right\}.$$

If the set is a subspace, find a basis.

3. Determine whether the subset $V \subset \mathbb{R}^2$ is a subspace, where

$$V = \left\{ \begin{bmatrix} -a + 1 \\ a - 2 \end{bmatrix} \quad \text{with} \quad a \in \mathbb{R} \right\}.$$

If the set is a subspace, find a basis.

4. Determine which of the following subsets of $M_{3,3}$ are subspaces. In the case that the set is a subspace, find a basis.
(a) The set of all symmetric matrices.
(b) The set of all skew-symmetric matrices.
(c) The set of al matrices A such that $A^2 = A$.
(d) The set of al matrices A such that $\operatorname{tr}(A) = 0$.
(e) The set of al matrices A such that $\det(A) = 0$.

5. (a) Find the dimension of both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 5 & 2 \\ 2 & 1 & 4 & 7 & 3 \\ 0 & -1 & 2 & -3 & -1 \end{bmatrix}.$$

(b) Is the linear transformation given by the matrix A one-to-one? Is it onto? Justify your answers and show your work.

6. (a) Find the dimension of both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 & 2 \\ 2 & 1 & 7 & 4 & 3 \\ 0 & -1 & -3 & 2 & -1 \end{bmatrix}.$$

(b) Is the linear transformation given by the matrix A one-to-one? Is it onto? Justify your answers and show your work.

7. Let $b = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $e = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be two bases of the vector space $\mathbb{R}^2$, where e is the standard basis. Let $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ be the matrix that transforms the components of a vector $\boldsymbol{x} \in \mathbb{R}^2$ from the basis e into the basis b, that is, $[\boldsymbol{x}]_b = P\,[\boldsymbol{x}]_e$. Find the components of the basis vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ in the standard basis, that is, find $[\boldsymbol{b}_1]_e$ and $[\boldsymbol{b}_2]_e$.

8. Let $b = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $e = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be two bases of the vector space $\mathbb{R}^2$, where e is the standard basis. Let $P = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ be the matrix that transforms the components of a vector $\boldsymbol{x} \in \mathbb{R}^2$ from the basis e into the basis b, that is, $[\boldsymbol{x}]_b = P\,[\boldsymbol{x}]_e$. Find the components of the basis vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ in the standard basis, that is, find $[\boldsymbol{b}_1]_e$ and $[\boldsymbol{b}_2]_e$.

9. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 2 & 0 \end{bmatrix}$.

(a) Verify that the vector $\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \\ -6 \\ 3 \end{bmatrix}$ belongs to the null space of A.
(b) Extend the set $\{\boldsymbol{v}\}$ into a basis of the null space of A.

10. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 1 & 3 & -3 \end{bmatrix}.$$

(a) Find a basis for the subspace of all vectors $\boldsymbol{b}$ such that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has solutions. Show your work.
(b) Find a basis for the null space of A. Show you work.

(c) Find a solution to the linear system $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$.

Is this solution unique? If yes, say why. If no, find a second solution $\boldsymbol{x}$ with the same $\boldsymbol{b}$.

4. Linear transformations

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 6 & -7 \\ -2 & 3 & 2 & 0 \\ 0 & -1 & -2 & 2 \end{bmatrix}$.

(a) Find a basis for $\mathrm{Col}(A)$, the column space of A.
(b) Find a basis for $N(A)$, the null space of A.
(c) Is the linear transformation determined by A one-to-one? Is it onto? Give reasons for your answers.

2. The matrices $A = \begin{bmatrix} 3 & 1 & 6 & -2 \\ 2 & 1 & 5 & 0 \\ 0 & -1 & -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are row equivalent.

(a) Find a basis for $\mathrm{Col}(A)$, the column space of A.
(b) Find a basis for $N(A)$, the null space of A.
(c) Is the linear transformation determined by A one-to-one? Is it onto? Give reasons for your answers.

3. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T(\boldsymbol{x}) = \begin{bmatrix} 2x_1 + 6x_2 - 2x_3 \\ 3x_1 + 8x_2 + 2x_3 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Find all solutions of the linear system $T(\boldsymbol{x}) = \boldsymbol{b}$, where $\boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and write these solutions in parametric form.

(b) Is the set of all solutions found in part (a) a subspace of $\mathbb{R}^3$? Give reasons for your answer.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathcal{S}_2}\right) = \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + x_2 \\ x_2 \end{bmatrix}_{\mathcal{S}_3}.$$

(a) Find the matrix $[T]_{\mathcal{S}_2\mathcal{S}_3}$ associated to the linear transformation T using the standard bases $\mathcal{S}_2$ and $\mathcal{S}_3$ of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.
(b) Is T one-to-one? Justify your answer.
(c) Is T onto? Justify your answer.

5. Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by

$$T(\boldsymbol{u}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad T(\boldsymbol{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(a) Find the matrix $A = [T(\boldsymbol{e}_1), T(\boldsymbol{e}_2)]$ of the linear transformation, where $\boldsymbol{e}_1 = \frac{1}{2}(\boldsymbol{u} + \boldsymbol{v})$ and $\boldsymbol{e}_2 = \frac{1}{2}(\boldsymbol{u} - \boldsymbol{v})$. Show your work.
(b) Compute the area of the parallelogram formed by $\boldsymbol{u}$ and $\boldsymbol{v}$. Compute also the area of the parallelogram formed by $T(\boldsymbol{u})$ and $T(\boldsymbol{v})$. Show your work.

6. Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathcal{S}_3}\right) = \begin{bmatrix} x_1 - 2x_2 + 3x_3 \\ -3x_1 + x_2 \end{bmatrix}_{\mathcal{S}_2}.$$

(a) Find the matrix A associated to the linear transformation T using the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^2$. Show your work.
(b) Find a basis for the column space of A. Show your work.
(c) Is T one-to-one? Is T onto? Justify your answer.

7. Consider the linear transformations $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathcal{S}}\right) = \begin{bmatrix} 2x_1 - x_2 + 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_2 + 3x_3 \end{bmatrix}_{\mathcal{S}}, \qquad S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathcal{S}}\right) = \begin{bmatrix} -x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix}_{\mathcal{S}},$$

where $\mathcal{S}$ is a standard basis of $\mathbb{R}^3$.
(a) Find a matrix $[T]_{\mathcal{SS}}$ and the matrix $[S]_{\mathcal{SS}}$. Show your work.
(b) Is T one-to-one? Is T onto? Justify your answer.
(c) Find the matrix of the composition $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ in the standard basis, that is, find $[T \circ S]_{\mathcal{SS}}$. Justify your answer.

5. Change of basis

1. Consider the vector space $\mathbb{R}^2$ and with bases

$$\mathcal{S} = \left\{ [\boldsymbol{e}_1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{S}}, [\boldsymbol{e}_2]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{S}} \right\}, \quad \mathcal{U} = \left\{ [\boldsymbol{u}_1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{S}}, [\boldsymbol{u}_2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{S}} \right\}.$$

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by

$$[T(\boldsymbol{u}_1)]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{\mathcal{S}}, \quad [T(\boldsymbol{u}_2)]_{\mathcal{S}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\mathcal{S}}.$$

Find the matrix $[T]_{\mathcal{US}}$, then the matrix $[T]_{\mathcal{SS}}$, and finally the matrix $[T]_{\mathcal{UU}}$, all matrices associated to the linear transformation above.

2. Let $\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ be a basis of $\mathbb{R}^2$ given by

$$\boldsymbol{u}_1 = 2\boldsymbol{e}_1 - 9\boldsymbol{e}_2, \quad \boldsymbol{u}_2 = \boldsymbol{e}_1 + 8\boldsymbol{e}_2,$$

where $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbb{R}^2$.
(a) Find both change of bases matrices $[I]_{\mathcal{US}}$ and $[I]_{\mathcal{SU}}$.
(b) Consider the vector $\boldsymbol{x} = 2\boldsymbol{u}_1 + \boldsymbol{u}_2$. Find both $[\boldsymbol{x}]_{\mathcal{S}}$ and $[\boldsymbol{x}]_{\mathcal{U}}$, that is, the components of $\boldsymbol{x}$ in the standard basis and in the $\mathcal{U}$ basis, respectively.

3. Let $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ and $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ be two bases of $\mathbb{R}^3$, and suppose that

$$\boldsymbol{c}_1 = \boldsymbol{b}_1 - 2\boldsymbol{b}_2 + \boldsymbol{b}_3, \quad \boldsymbol{c}_2 = -\boldsymbol{b}_2 + 3\boldsymbol{b}_3, \quad \boldsymbol{c}_3 = -2\boldsymbol{b}_1 + \boldsymbol{b}_3.$$

(a) Find the change of basis matrices $[I]_{\mathcal{BC}}$ and $[I]_{\mathcal{CB}}$.
(b) Consider the vector $\boldsymbol{x} = \boldsymbol{c}_1 - 2\boldsymbol{c}_2 + 2\boldsymbol{c}_3$. Find both $[\boldsymbol{x}]_{\mathcal{B}}$ and $[\boldsymbol{x}]_{\mathcal{C}}$, that is, the components of $\boldsymbol{x}$ in the bases $\mathcal{B}$ and $\mathcal{C}$, respectively.

4. Consider the bases $\mathcal{U}$ and $\mathcal{S}$ of $\mathbb{R}^2$ given by

$$\left\{ [\boldsymbol{u}_1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{S}}, [\boldsymbol{u}_2]_{\mathcal{S}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{S}} \right\}, \quad \mathcal{S} = \left\{ [\boldsymbol{e}_1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{S}}, [\boldsymbol{e}_2]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{S}} \right\}.$$

(a) Given $[\boldsymbol{x}]_{\mathcal{U}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{\mathcal{U}}$ find $[\boldsymbol{x}]_{\mathcal{S}}$, that is, the components of the vector $\boldsymbol{x}$ in the standard basis of $\mathbb{R}^2$.

(b) Find the components of the standard basis in terms of the $\mathcal{U}$ basis, that is, find $[\boldsymbol{e}_1]_{\mathcal{U}}$ and $[\boldsymbol{e}_2]_{\mathcal{U}}$.

5. Let $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $\mathcal{S} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be two bases of the vector space $\mathbb{R}^2$, where $\mathcal{S}$ is the standard basis. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ be the matrix that transforms the components of a vector $\boldsymbol{x} \in \mathbb{R}^2$ from the basis $\mathcal{S}$ into the basis $\mathcal{B}$, that is, $[\boldsymbol{x}]_{\mathcal{B}} = A\,[\boldsymbol{x}]_{\mathcal{S}}$. Find the components of the basis vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ in the standard basis, that is, find $[\boldsymbol{b}_1]_{\mathcal{S}}$ and $[\boldsymbol{b}_2]_{\mathcal{S}}$.

6. Let $\mathcal{S}_3$ and $\mathcal{S}_2$ be standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, and consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathcal{S}_3}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_2 - x_3 \end{bmatrix}_{\mathcal{S}_2},$$

and introduce the bases

$$\mathcal{U} = \left\{ [\boldsymbol{u}_1]_{\mathcal{S}_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{S}_3}, [\boldsymbol{u}_2]_{\mathcal{S}_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{S}_3}, [\boldsymbol{u}_3]_{\mathcal{S}_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{S}_3} \right\} \subset \mathbb{R}^3,$$

$$\mathcal{V} = \left\{ [\boldsymbol{v}_1]_{\mathcal{S}_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{S}_2}, [\boldsymbol{v}_2]_{\mathcal{S}_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{S}_2} \right\} \subset \mathbb{R}^2.$$

Find the matrices $[T]_{\mathcal{S}_3\mathcal{S}_2}$ and $[T]_{\mathcal{U}\mathcal{V}}$.

6. Inner product

1. Let V be an inner product space, with inner product $\langle\,,\rangle$, and let $\boldsymbol{x}$, $\boldsymbol{y} \in V$. Show that $\boldsymbol{x} - \boldsymbol{y}$ is orthogonal to $\boldsymbol{x} + \boldsymbol{y}$ iff $\|\boldsymbol{x}\| = \|\boldsymbol{y}\|$.

2. Consider the subspace $W = \operatorname{Span}\left\{ \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

(a) Find an orthogonal decomposition of the vector $\boldsymbol{u}_2$ with respect to the vector $\boldsymbol{u}_1$. Using this decomposition, find an orthogonal basis for the space W.

(b) Find the decomposition of the vector $\boldsymbol{y} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ in orthogonal components with respect to the subspace W.

(c) Find the vector in W which is the closest vector to $\boldsymbol{y}$.

3. Find all vectors in $\mathbb{R}^4$ perpendicular to both $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 9 \\ 8 \\ 2 \end{bmatrix}$.

4. Consider the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ -2 & 0 \end{bmatrix}$ and the vector $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(a) Find the best approximation (least-squares) solution $\widehat{\boldsymbol{x}}$ to the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$.

(b) Find the orthogonal projection of the source vector $\boldsymbol{b}$ onto the subspace $\operatorname{Col}(A)$, the column space of A.

5. Given the matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, which has linearly independent column vectors, find an orthonormal basis for the space Col(A), the column space of A, using the Gram-Schmidt process starting with the first column vector.

6. Consider the subspace $W \subset \mathbb{R}^3$ given by $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

(a) Find a basis for the space $W^\perp$, that is, find a basis for the orthogonal complement of the space W.
(b) Use the result in part (6a) to find an orthogonal basis for the same space $W^\perp$.

7. Consider the subspace $W \subset \mathbb{R}^3$ given by $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$.

(a) Find a basis for the space $W^\perp$, that is, find a basis for the orthogonal complement of the space W.
(b) Use the result in part (7a) to find an orthogonal basis for the same space $W^\perp$.

8. Given the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$, find a basis for the space Col$(A)^\perp$, the orthogonal complement of the column space of A.

9. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -2 & 0 \end{bmatrix}$ and the vector $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(a) Find the best approximation (least-squares) solution $\hat{\boldsymbol{x}}$ to the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$.
(b) Verify that the vector $A\hat{\boldsymbol{x}} - \boldsymbol{b}$, where $\hat{\boldsymbol{x}}$ is the least-squares solution found in part (9a), belongs to the space Col$(A)^\perp$, the orthogonal complement of the column space of A.

10. Let V be a vector space with inner product $\langle\, , \rangle$, and associated norm $\|\ \|$. Let $\boldsymbol{x}$, $\boldsymbol{y} \in V$, where $\boldsymbol{x}$ is an eigenvector of a matrix A with eigenvalue 2, and $\boldsymbol{y}$ is another eigenvector with eigenvalue -3. Assume that $\|\boldsymbol{x}\| = 1/3$, $\|\boldsymbol{y}\| = 1$ and $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.
(a) Compute $\|\boldsymbol{v}\|$ for $\boldsymbol{v} = 3\boldsymbol{x} - \boldsymbol{y}$.
(b) Compute $\|A\boldsymbol{v}\|$ for the $\boldsymbol{v}$ given above.

11. (a) Find a basis for both the null space and the column space of A, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -4 \\ 0 & 1 & 2 \end{bmatrix}.$$

(b) Find Col$(A)^\perp$.
(c) Find an orthogonal basis for the column space of A.

12. Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\left\{ \boldsymbol{u}_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\},$$

using the Gram-Schmidt process starting with the vector $\boldsymbol{u}_1$.

13. Consider the subspace $W \subset \mathbb{R}^3$ given by

$$W = \text{Span}\left\{ \boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

(a) Find an ortho**normal** basis of W using the Gram-Schmidt process starting with the vector $\boldsymbol{u}_1$. Show your work.

(b) Decompose the vector $\boldsymbol{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ as follows, $\boldsymbol{x} = \hat{\boldsymbol{x}} + \boldsymbol{x}'$, with $\hat{\boldsymbol{x}} \in W$ and $\boldsymbol{x}'$ perpendicular to any vector in W. Show your work.

14. Find the third column in matrix Q below such that $Q^T = Q^{-1}$, where

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & Q_{13} \\ 1/\sqrt{3} & 2/\sqrt{14} & Q_{23} \\ 1/\sqrt{3} & -3/\sqrt{14} & Q_{33} \end{bmatrix}$$

15. (a) If Q_1 and Q_2 are orthogonal matrices, that is, each matrix satisfies $Q^T = Q^{-1}$, then show that $Q_1 Q_2$ is also orthogonal.

(b) If Q_1 is a counterclockwise rotation of an angle θ and Q_2 is a counterclockwise rotation by an angle ϕ, what is $Q_1 Q_2$?

16. If the vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2 \in \mathbb{R}^n$ form an orthonormal set, what combination of them is the closest to a vector $\boldsymbol{b} \in \mathbb{R}^n$? Verify that the error is orthogonal to $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

17. Find the QR factorization of matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

18. Use Gram-Schmidt method on the columns of matrix A to find its QR factorization, where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

7. Eigenvalues and Eigenvectors

1. (a) Find all the eigenvalues and eigenvectors of the matrix, $A = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$.

(b) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

2. (a) Find all the eigenvalues and their corresponding algebraic multiplicities of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & h \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) Find the value(s) of the real number h such that the matrix A above has a two-dimensional eigenspace, and find a basis for this eigenspace.

(c) Set $h = 1$, and find a basis for all the eigenspaces of matrix A above.

3. Let A be a 4×4 matrix that can be decomposed as $A = PDP^{-1}$, with P an invertible matrix and the matrix $D = \text{diag}(2, \frac{1}{4}, 2, 3)$. Knowing only this information about the matrix A, is it possible to compute the $\det(A)$? If your answer is ***no***, explain why not; if your answer is ***yes***, compute $\det(A)$ and show your work.

4. Let A be a 4×4 matrix that can be decomposed as $A = PDP^{-1}$, with P an invertible matrix and the matrix $D = \text{diag}(2, 0, 2, 5)$. Knowing only this information about the matrix A, is it possible to whether A invertible? Is it possible to know $\text{tr}\,(A)$? If your answer is ***no***, explain why not; if your answer is ***yes***, compute $\det(A)$ and show your work.

5. Comparing the characteristic polynomials for A and A^T, show that these two matrices have the same eigenvalues.

6. Which of the following matrices cannot be diagonalized?

$$A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}.$$

7. (a) Find all the eigenvalues with their corresponding algebraic multiplicities, and find all the associated eigenspaces of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Is matrix A diagonalizable? If your answer is ***yes***, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If your answer is ***no***, explain why.

8. Let A be a 3×3 matrix with eigenvalues 2, -1 and 3.
(a) Find the eigenvalues of A^{-1}.
(b) Find the determinant of A.
(c) Find the determinant of A^{-1}.
(d) Find the eigenvalues of $A^2 - A$.

9. Let k be any number, and consider the matrix A given by

$$A = \begin{bmatrix} 2 & -2 & 4 & -1 \\ 0 & 3 & k & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A, and their corresponding multiplicity. Show your work.
(b) Find the number k such that there exists an eigenspace $E_A(\lambda)$ that is two dimensional, and find a basis for this $E_A(\lambda)$. The notation $E_A(\lambda)$ means the eigenspace corresponding to the eigenvalue λ of matrix A. Show your work.

10. (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Show your work.
(b) Find matrices P and D such that $A = PDP^{-1}$, where P is invertible an D diagonal. Show your work.
(c) Compute A^5.

11. If A is diagonalizable, show that $\det(A)$ is the product of its eigenvalues.

12. Suppose that a 3×3 matrix A has eigenvalues $1, 2, 4$. What is the trace of A? What is the trace of A^2? What is $\det[(A^{-1})^T]$?