Name: ________________________________ **PID Number:** ____________________

MTH 415
Practice Final Exam
August 17, 2009

- *No calculators or any other devices are allowed on this exam.*
- *Read each question carefully. If any question is not clear, ask for clarification.*
- *Write your solutions clearly and legibly; no credit will be given for illegible solutions.*
- *Answer each question completely, and show all your work.*
- *If you present different answers, then the worst answer will be graded.*

Signature: ____

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Σ	200	

1. (20 points) Consider the matrix $\mathsf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$. Find the coefficients $(\mathsf{A}^{-1})_{21}$ and $(\mathsf{A}^{-1})_{32}$ of the matrix A^{-1}, that is, of the inverse matrix of A. Show your work.

2. (20 points)

(a) Find $k \in \mathbb{R}$ such that the volume of the parallelepiped formed by the vectors below is equal to 4, where

$$\mathsf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathsf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathsf{v}_3 = \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix}$$

(b) Set $k = 1$ and define the matrix $\mathsf{A} = [\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3]$. Matrix A determines the linear transformation $\mathsf{A} : \mathbb{R}^3 \to \mathbb{R}^3$. Is this linear transformation injective (one-to-one)? Is it surjective (onto)?

3. Consider the matrix $\mathsf{A} = \begin{bmatrix} -1/2 & -3 \\ 1/2 & 2 \end{bmatrix}$.

(a) (10 points) Show that matrix A is diagonalizable.

(b) (10 points) Using that A is diagonalizable, find the $\lim_{k\to\infty} \mathsf{A}^k$.

4. (20 points) Determine whether the subset $V \subset \mathbb{R}^3$ is a subspace, where

$$V = \left\{ \begin{bmatrix} -a+b \\ a-2b \\ a-7b \end{bmatrix} \quad \text{with} \quad a, b \in \mathbb{R} \right\}.$$

If the set is a subspace, find an orthogonal basis in the inner product space $\left(\mathbb{R}^3, \cdot\right)$.

5. True of false: (Justify your answers.)

(a) (10 points) If the set of columns of $\mathsf{A} \in \mathbb{F}^{m,n}$ is a linearly independent set, then $\mathsf{Ax} = \mathsf{b}$ has exactly one solution for every $\mathsf{b} \in \mathbb{F}^m$.

(b) (10 points) The set of column vectors of an 5×7 is never linearly independent.

6. (20 points) Consider the linear transformations $\boldsymbol{T} : \mathbb{R}^3 \to \mathbb{R}^2$ and $\boldsymbol{S} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\left[\boldsymbol{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{s_3}\right)\right]_{s_2} = \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + 2x_2 + x_3 \end{bmatrix}_{s_2}, \qquad \left[\boldsymbol{S}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{s_3}\right)\right]_{s_3} = \begin{bmatrix} 3x_3 \\ 2x_2 \\ x_1 \end{bmatrix}_{s_3},$$

where $\mathcal{S}_3$ and $\mathcal{S}_2$ are the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

(a) Find a matrix $[\boldsymbol{T}]_{s_3 s_2}$ and the matrix $[\boldsymbol{S}]_{s_3 s_3}$. Show your work.

(b) Find the matrix of the composition $\boldsymbol{T} \circ \boldsymbol{S} : \mathbb{R}^3 \to \mathbb{R}^2$ in the standard basis, that is, find $[\boldsymbol{T} \circ \boldsymbol{S}]_{s_3 s_2}$.

(c) Is $\boldsymbol{T} \circ \boldsymbol{S}$ injective (one-to-one)? Is $\boldsymbol{T} \circ \boldsymbol{S}$ surjective (onto)? Justify your answer.

7. (20 points) Consider the vector space $\mathbb{R}^2$ with the standard basis $\mathcal{S}$ and let $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$[\boldsymbol{T}]_{ss} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}_{ss}.$$

Find $[\boldsymbol{T}]_{bb}$, the matrix of $\boldsymbol{T}$ in the basis $\mathcal{B}$, where $\mathcal{B} = \left\{ [\mathsf{b}_1]_s = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_s, [\mathsf{b}_2]_s = \begin{bmatrix} 1 \\ -2 \end{bmatrix}_s \right\}.$

8. Let $\big(V, \langle\,,\rangle\big)$ be an inner product space with inner product norm $\|\ \|$. Let $\boldsymbol{T}: V \to V$ be a linear transformation and $\boldsymbol{x},\ \boldsymbol{y} \in V$ be vectors satisfying the following conditions:

$$\boldsymbol{T}(\boldsymbol{x}) = 2\,\boldsymbol{x}, \quad \boldsymbol{T}(\boldsymbol{y}) = -3\,\boldsymbol{y}, \quad \|\boldsymbol{x}\| = 1/3, \quad \|\boldsymbol{y}\| = 1, \quad \boldsymbol{x} \perp \boldsymbol{y}.$$

(a) (10 points) Compute $\|\boldsymbol{v}\|$ for the vector $\boldsymbol{v} = 3\boldsymbol{x} - \boldsymbol{y}$.

(b) (10 points) Compute $\|\boldsymbol{T}(\boldsymbol{v})\|$ for the vector $\boldsymbol{v}$ given above.

9. (20 points) Consider the matrix $\mathsf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$ and the vector $\mathsf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find the least-squares solution $\hat{\mathsf{x}}$ to the matrix equation $\mathsf{A}\,\mathsf{x} = \mathsf{b}$.

(b) Verify whether the vector $\mathsf{A}\,\hat{\mathsf{x}} - \mathsf{b}$ belong to the space $R(\mathsf{A})^{\perp}$? Justify your answers.

10. Consider the matrix $\mathsf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & h \\ 0 & 0 & 2 \end{bmatrix}$.

(a) (10 points) Find all eigenvalues of matrix A and their corresponding algebraic multiplicities.

(b) (10 points) Find the value(s) of the real number h such that the matrix A above has a two-dimensional eigenspace, and find a basis for this eigenspace.