Name: Key PID Number: ____________

MTH 415
Final Exam
August 19, 2009

- *No calculators or any other devices are allowed on this exam.*
- *Read each question carefully. If any question is not clear, ask for clarification.*
- *Write your solutions clearly and legibly; no credit will be given for illegible solutions.*
- *Answer each question completely, and show all your work.*
- *If you present different answers, then the worst answer will be graded.*

Signature: ____________________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Σ	200	

1. (20 points) Consider the matrix $A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$. Find the coefficients $(A^{-1})_{13}$ and $(A^{-1})_{21}$ of the inverse matrix of A. Show your work.

$$(A^{-1})_{13} = \frac{C_{31}}{\det(A)}$$

$$\det(A) = (-2)(2-1) - 3(1-2) - 1(-1+4)$$

$$= -2 + 3 - 3$$

$$\boxed{\det(A) = -2}$$

$$\boxed{(A^{-1})_{13} = \frac{1}{2}}$$

$$C_{31} = (-1)^4 \det(\mathring{A}_{31})$$

$$= \begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} = -3 + 2 = -1$$

$$(A^{-1})_{21} = \frac{C_{12}}{\det(A)}$$

$$C_{12} = (-1) \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = -(1-2) = 1$$

$$\boxed{(A^{-1})_{21} = -\frac{1}{2}}$$

2. (20 points) Consider the vector space $\mathbb{P}_3([0,1])$ with the inner product

$$\langle \boldsymbol{p}, \boldsymbol{q} \rangle = \int_0^1 \boldsymbol{p}(x)\boldsymbol{q}(x)\,dx.$$

Given the set $\mathcal{U} = \{\boldsymbol{p}_1 = x^2,\ \boldsymbol{p}_2 = x^3\}$, find an orthogonal basis for the subspace $U = \mathrm{Span}(\mathcal{U})$ using the Gram-Schmidt method on the set $\mathcal{U}$ starting with the vector $\boldsymbol{p}_1$.

$$\boxed{q_1 = x^2} \qquad \|q_1\|^2 = \int_0^1 x^4\,dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$$

$$q_2 = x^3 - \frac{\langle q_1, p_2 \rangle}{\|q_1\|^2} x^2$$

$$\langle q_1, p_2 \rangle = \int_0^1 x^5\,dx = \frac{x^6}{6}\Big|_0^1 = \frac{1}{6}$$

$$q_2 = x^3 - \frac{1}{6}\,5\,x^2$$

$$\boxed{q_2 = x^3 - \frac{5}{6}x^2}$$

3. Consider the matrix $A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 3 & 0 \\ 3 & 9 & 5 & -1 \end{bmatrix}$.

(a) (10 points) Verify that the vector $v = \begin{bmatrix} 3 \\ 1 \\ -4 \\ -2 \end{bmatrix}$ belongs to the null space of A.

(b) (10 points) Extend the set $\{v\}$ into a basis of the null space of A.

(a)
$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 3 & 0 \\ 3 & 9 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+3-4-2 \\ 6+6-12+0 \\ 9+9-20+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = -3x_2 - 3x_4 \\ x_3 = 2x_4 \end{array}$$

$$X = \begin{bmatrix} -3x_2 - 3x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} x_4$$

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} \qquad \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ -4 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

$$\boxed{\text{Basis } \left\{ \begin{bmatrix} 3 \\ 1 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$$

4. (20 points) Use Cramer's rule to find the solution to the linear system

$$2x_1 + x_2 - x_3 = 0$$
$$x_1 + x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 0.$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -(3-1) - 2(2+1) = -2 - 6$$

$$\boxed{\det(A) = -8}$$

$$x_1 = \frac{\begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix}}{(-8)} = \frac{-(3+2)}{(-8)} = 1 \Rightarrow \boxed{x_1 = \frac{5}{8}}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix}}{(-8)} = \frac{(6+1)}{-8} \Rightarrow \boxed{x_2 = -\frac{7}{8}}$$

$$x_3 = \frac{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix}}{(-8)} = \frac{-(4-1)}{(-8)} = \frac{3}{5} \Rightarrow \boxed{x_3 = \frac{3}{8}}$$

$$\boxed{\underline{x} = \frac{1}{8} \begin{bmatrix} 5 \\ -7 \\ 3 \end{bmatrix}}$$

5. (20 points) Let $\mathcal{S}_3$ and $\mathcal{S}_2$ be standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, and consider the linear transformation $\boldsymbol{T}\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\left[\boldsymbol{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{s3}\right)\right]_{s2} = \begin{bmatrix} -x_1 + 2x_2 - x_3 \\ x_1 + x_3 \end{bmatrix}_{s2},$$

and introduce the bases $\mathcal{U} \subset \mathbb{R}^3$ and $\mathcal{V} \subset \mathbb{R}^2$ given by

$$\mathcal{U} = \left\{ [\mathsf{u}_1]_{s3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{s3},\ [\mathsf{u}_2]_{s3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{s3},\ [\mathsf{u}_3]_{s3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{s3} \right\},$$

$$\mathcal{V} = \left\{ [\mathsf{v}_1]_{s2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{s2},\ [\mathsf{v}_2]_{s2} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}_{s2} \right\}.$$

Find the matrices $[\boldsymbol{T}]_{s3s2}$ and $[\boldsymbol{T}]_{uv}$. Show your work.

$$P = I_{us} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$T_{ss} = \begin{bmatrix} -1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$Q = I_{vs} = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$$

$$T_{uv} = Q^{-1}\, T_{ss}\, P$$

$$Q^{-1} = \frac{1}{(2+6)} \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\boxed{T_{uu} = \frac{1}{8} \begin{bmatrix} 5 & 2 & 5 \\ -1 & 6 & -1 \end{bmatrix}}$$

6. Consider the inner product space $(\mathbb{R}^{2,2}, \langle\,,\rangle_F)$ and the subspace

$$W = \mathrm{Span}\left\{\mathsf{E}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathsf{E}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\} \subset \mathbb{R}^{2,2}.$$

Find a basis for $W^\perp$, the orthogonal complement of W.

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \qquad 0 = \langle E_1, X \rangle = tr\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = x_{21} + x_{12}$$

$$0 = \langle E_2, X \rangle = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = x_{11} - x_{22}$$

$$\boxed{x_{21} = -x_{12}} \qquad \boxed{x_{11} = x_{22}}$$

$$X = \begin{bmatrix} x_{11} & x_{12} \\ -x_{12} & x_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_{11} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_{12}.$$

$$\boxed{W^\perp = \mathrm{Span}\Bigg(\underbrace{\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}}_{\text{basis of } W^\perp .}\Bigg)}$$

7. (20 points) Consider the matrix $\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \end{bmatrix}$ and the vector $\mathsf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) Find the least-squares solution $\hat{\mathsf{x}}$ to the matrix equation $\mathsf{A}\,\mathsf{x} = \mathsf{b}$.

(b) Verify that the vector $\mathsf{A}\,\hat{\mathsf{x}} - \mathsf{b}$, where $\hat{\mathsf{x}}$ is the least-squares solution found in part (a), belongs to the space $R(\mathsf{A})^{\perp}$, the orthogonal complement of $R(\mathsf{A})$.

(a) $$A^T A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\hat{x} = \frac{1}{35} \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 5 \\ -5 \end{bmatrix} \Rightarrow \boxed{\hat{x} = \frac{1}{7} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

(b) $$A\hat{x} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

$$A\hat{x} - b = \frac{1}{7}\left(\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} \right) \Rightarrow \boxed{(A\hat{x} - b) = \frac{1}{7} \begin{bmatrix} -1 \\ -5 \\ -3 \end{bmatrix}}$$

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = 1 + 5 - 6 = 0 \checkmark$$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = 2 - 5 + 3 = 0 \checkmark$$

8. Suppose that a matrix $A \in \mathbb{R}^{3,3}$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 4$.

(a) (15 points) Find the trace of A, find the trace of A^2, and find the determinant of A.

(b) (5 points) Is matrix A invertible? If your answer is "yes", then prove it and find $\det(A^{-1})$; if your answer is "no", then prove it.

$A \in \mathbb{R}^{3,3}$ $\quad \left.\begin{array}{l} \lambda_1 \neq \lambda_2 \\ \quad \neq \lambda_3 \\ \lambda_2 \neq \lambda_3 \end{array}\right\} \Rightarrow \{v_1, v_2, v_3\}$ l.i. , where $A v_i = \lambda_i v_i$, $i = 1, 2, 3$.

$\therefore$ A diagonalizable $\Rightarrow \exists P, P^{-1}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ s.t.

$$\boxed{A = PDP^{-1}}$$

(a) $\left[tr(A) = tr(PDP^{-1}) = tr(P^{-1}PD) = tr(D) = 1+2+4 = 7 \right]$

$\left[tr(A^2) = tr(PD^2P^{-1}) = tr(D^2) = 1+4+16 = 21 \right]$

$\left[\det(A) = \det(PDP^{-1}) = \det(D) = (1)(2)(4) = 8. \right]$

(b) Set of columns of A is l.i. $\Rightarrow$ A invertible.

$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1} D^{-1} P^{-1} = P D^{-1} P^{-1}$

$\det(A^{-1}) = \det(D^{-1}) = \dfrac{1}{8}$.

9. Consider the matrix $A = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}$.

(a) (10 points) Find the eigenvalues and eigenvectors of A.

(b) (10 points) Compute the matrix e^{A}.

(a) $\begin{vmatrix} 7-\lambda & 5 \\ 3 & -7-\lambda \end{vmatrix} = (\lambda-7)(\lambda+7) - 15 = \lambda^2 - 49 - 15$

$= \lambda^2 - 64$

$\lambda^2 - 64 = 0 \implies \boxed{\lambda_\pm = \pm 8}$

λ_+ $\begin{bmatrix} -1 & 5 \\ 3 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$ $x_1 = 5x_2$ $\boxed{\underline{x}_+ = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \lambda_+ = 8}$

λ_- $\begin{bmatrix} 15 & 5 \\ 3 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ $3x_1 = -x_2$ $\boxed{\underline{x}_- = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \lambda_- = -8}$

(b) $e^A = P e^D P^{-1}$ $P = \begin{bmatrix} +5 & 1 \\ 1 & -3 \end{bmatrix}$ $D = \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}$

$P^{-1} = \frac{(-1)}{16} \begin{bmatrix} -3 & -1 \\ -1 & +5 \end{bmatrix}$

$$\boxed{e^A = \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} e^8 & 0 \\ 0 & e^{-8} \end{bmatrix} \frac{(-1)}{16} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix}}$$

10. (20 points) Find the function $\mathsf{x} : \mathbb{R} \to \mathbb{R}^2$ solution of the initial value problem

$$\frac{d}{dt}\mathsf{x}(t) = \mathsf{A}\,\mathsf{x}(t), \qquad \mathsf{x}(0) = \mathsf{x}_0,$$

where the matrix $\mathsf{A} = \begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}$ and the vector $\mathsf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{vmatrix} -5-\lambda & 2 \\ -12 & 5-\lambda \end{vmatrix} = (\lambda-5)(\lambda+5) + 24 = \lambda^2 - 25 + 24 = \lambda^2 - 1$$

$$\boxed{\lambda_\pm = \pm 1}$$

$$\boxed{\lambda_+ = 1} \quad \begin{bmatrix} -6 & 2 \\ -12 & 4 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \quad +3x_1 = x_2$$

$$\boxed{\underline{x}_+ = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$$\lambda_- = -1 \quad \begin{bmatrix} -4 & 2 \\ -12 & 6 \end{bmatrix} \to \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \quad 2x_1 = x_2 \quad \boxed{\underline{x}_- = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \lambda_- = -1}$$

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2-3)} \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= (-1) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

$$\boxed{\underline{x}(t) = -e^{t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$