Name: ________________________________ **ID Number:** ____________________

MTH 415
Exam 1
February 04, 2009

Read each question carefully. If any question is not clear, ask for clarification.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Answer each question completely, and show all your work.

1. (20 points) Find the general solution to the homogeneous linear system with coefficient matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 5 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix},$$

and write this general solution in vector form.

#	Score
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Solution Problem 1:

We use Gauss-Jordan's method to find the general solution to the system $A\boldsymbol{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 3 & -1 & 5 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & -7 & 7 & -14 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore, the solution is

$$\begin{aligned} x_1 &= -2x_3 + x_4 \\ x_2 &= x_3 - 2x_4 \\ x_3 &: \text{free} \\ x_4 &: \text{free.} \end{aligned} \quad \Rightarrow \quad \boxed{\boldsymbol{x} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} x_4}.$$

2. (a) (10 points) Find a value of the constants h and k such that the non-homogeneous linear system below is consistent and has one free variable.

$$\begin{aligned} x_1 + h\,x_2 + 5x_3 &= 1, \\ x_2 - 2x_3 &= k, \\ x_1 + 3x_2 - 3x_3 &= 5. \end{aligned}$$

(b) (10 points) Using the value of the constants h and k found in part (2a), find the general solution to the system given in part (2a).

Solution Problem 2:

(a)

$$\left[\begin{array}{ccc|c} 1 & h & 5 & 1 \\ 0 & 1 & -2 & k \\ 1 & 3 & -3 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & h & 5 & 1 \\ 0 & 1 & -2 & k \\ 0 & 3-h & -8 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & h & 5 & 1 \\ 0 & 1 & -2 & k \\ 0 & 0 & -2-2h & 4-(3-h)k \end{array}\right].$$

Therefore,

$$2 + 2h = 0 \quad \Rightarrow \quad \boxed{h = -1},$$

$$4 - (3-h)k = 0, \quad h = -1 \quad \Rightarrow \quad 4 - 4k = 0 \quad \Rightarrow \quad \boxed{k = 1}.$$

(b)

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} x_1 = 2 - 3x_3 \\ x_2 = 1 + 2x_3 \\ x_3 : \text{ free.} \end{array}$$

Therefore, the solution is

$$\boxed{\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} x_3}.$$

3. Consider the linear system

$$\begin{aligned} 6x_1 + 9x_2 &= 33, \\ 7x_1 + 3x_2 &= 16. \end{aligned}$$

(a) (10 points) Use 3-digit arithmetic, **no** pivoting and **no** scaling, to find the solution of the system above.

(b) (10 points) Use 3-digit arithmetic, **with** partial pivoting and **no** scaling to find the solution of the system above.

Solution Problem 3:

(a)

$$\left[\begin{array}{cc|c} 6 & 9 & 33 \\ 7 & 3 & 16 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & 9 & 33 \\ 0.02 & 7.5 & 22.6 \end{array}\right]$$

since $f_\ell(7/6) = f_\ell(1.1666...) = 1.17$, and so:

$$\begin{aligned} f_\ell(6 \times 1.17) &= 7.02, \\ f_\ell(9 \times 1.17) &= f_\ell(10.53) = 10.5, \\ f_\ell(33 \times 1.17) &= f_\ell(38.61) = 38.6. \end{aligned}$$

We now modify the Gauss method:

$$\left[\begin{array}{cc|c} 6 & 9 & 33 \\ 0.02 & 7.5 & 22.6 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & 9 & 33 \\ 0 & 7.5 & 22.6 \end{array}\right].$$

The next step is: $f_\ell(9/7.5) = f_\ell(1.2) = 1.2$, and so:

$$\begin{aligned} f_\ell(7.5 \times 1.2) &= f_\ell(9) = 9, \\ f_\ell(22.6 \times 1.2) &= f_\ell(27.12) = 27.1, \end{aligned}$$

so

$$\left[\begin{array}{cc|c} 6 & 9 & 33 \\ 0 & 7.5 & 22.6 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & 0 & 5.9 \\ 0 & 7.5 & 22.6 \end{array}\right].$$

We now multiply each row by the following, respectively,

$$\begin{aligned} f_\ell(1/6) &= f_\ell(0.1666...) = 0.167, \\ f_\ell(1/7.5) &= f_\ell(1.333...) = 0.133, \end{aligned}$$

and the result is

$$\begin{aligned} f_\ell(6 \times 0.167) &= f_\ell(1.002) = 1, \\ f_\ell(7.5 \times 0.133) &= f_\ell(0.9975) = 1, \\ f_\ell(5.9 \times 0.167) &= f_\ell(0.9853) = 0.985, \\ f_\ell(22.6 \times 0.133) &= f_\ell(3.0058) = 3.01. \end{aligned}$$

So the result is

$$\boxed{x_1 = 0.985, \quad x_2 = 3.01}.$$

(b) Partial pivoting means,

$$\left[\begin{array}{cc|c} 6 & 9 & 33 \\ 7 & 3 & 16 \end{array}\right] \to \left[\begin{array}{cc|c} 7 & 3 & 16 \\ 6 & 9 & 33 \end{array}\right].$$

We now proceed as above: since $f_\ell(6/7) = f_\ell(0.8571...) = 0.857$, we obtain,

$$\left[\begin{array}{cc|c} 7 & 3 & 16 \\ 6 & 9 & 33 \end{array}\right] \to \left[\begin{array}{cc|c} 7 & 3 & 16 \\ 0 & 6.43 & 19.3 \end{array}\right].$$

Now, $f\ell(3/6.43) = f\ell(0.46656...) = 0.467$, we obtain,

$$\left[\begin{array}{cc|c} 7 & 3 & 16 \\ 0 & 6.43 & 19.3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 7 & 0 & 6.99 \\ 0 & 6.43 & 19.3 \end{array}\right].$$

We now multiply each row by the following, respectively,

$$\begin{aligned} f\ell(1/7) &= f\ell(0.14285...) = 0.143, \\ f\ell(1/6.43) &= f\ell(0.15552...) = 0.156, \end{aligned}$$

and the result is

$$\begin{aligned} f\ell(7 \times 0.143) &= f\ell(1.001) = 1, \\ f\ell(6.43 \times 0.156) &= f\ell(1.00308) = 1, \\ f\ell(6.99 \times 0.143) &= f\ell(0.99957) = 1, \\ f\ell(22.6 \times 0.156) &= f\ell(3.0108) = 3.01. \end{aligned}$$

So the result is

$$\boxed{x_1 = 1, \quad x_2 = 3.01}.$$

(A calculation in a different order gives $x_2 = 3$, which is also taken as correct.)

4. (20 points) Find the rank of matrix A below, and write the non-basic columns of A as a combination of its basic columns, where

$$A = \begin{bmatrix} 2 & -4 & -8 & 6 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 0 & 0 \end{bmatrix}.$$

(Recall: A column of matrix A is basic if E_A has a pivot in that column.)

Solution Problem 4:

$$\begin{bmatrix} 2 & -4 & -8 & 6 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & 3 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 4 & 12 & -9 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -17 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, $\boxed{r = 3}$ and

$$\boxed{\begin{bmatrix} -8 \\ 3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}}.$$

5. (a) (13 points) Find the general solution to the system below and write it in vector form,

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2, \\ 3x_1 + 7x_2 - 3x_3 &= 7, \\ x_1 + 4x_2 - x_3 &= 4. \end{aligned}$$

(b) (7 points) Sketch a graph on $\mathbb{R}^3$ of the general solution found in part (5a).

Solution Problem 5:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 3 & 7 & -3 & 7 \\ 1 & 4 & -1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} x_1 = x_3 \\ x_2 = 1 \\ x_3 : \text{ free.} \end{array}$$

So, the solution is the line

$$\boxed{\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}.$$