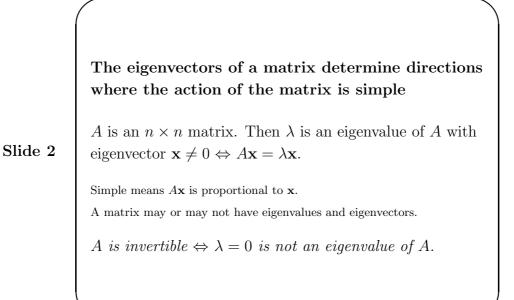
Slide 1

On determinants and eigenvalues

- Review: eigenvalues and eigenvectors.
- Eigenspaces.
- Characteristic equation.
- Multiplicity of eigenvalues.



Examples of eigenvalues and eigenvectors • The matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has eigenvectors and eigenvalues $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 4, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda_2 = -2.$ • The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has eigenvectors and eigenvalues $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \lambda_1 = 0, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \lambda_2 = 7.$

Eigenspaces are the subspaces spanned by the
eigenvectorsDefinition 1 Let λ be an eigenvalue of A. The
eigenspace $E_A(\lambda)$ is the set of all vectors \mathbf{x} solutions of
 $A\mathbf{x} = \lambda \mathbf{x}$.Slide 4Theorem 1 If λ is an eigenvector of an $n \times n$ matrix A,
then the set $E_A(\lambda) \subset \mathbb{R}^n$ is a subspace.Eigenspaces are lines, planes, or hyperplanes
through the origin

Here are the eigenspaces of the previous examples The matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has eigenvectors $\lambda_1 = 4$ and $\lambda_2 = -2$. The corresponding eigenspaces are Slide 5 $E(4) = \left\{ \mathbf{x} = t \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \ t \in \mathbb{R} \right\},$ $E(-2) = \left\{ \mathbf{x} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ t \in \mathbb{R} \right\}.$

Here are the eigenspaces of the previous examples The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has eigenvectors $\lambda_1 = 0$ and $\lambda_2 = 7$. The corresponding eigenspaces are $E(0) = \left\{ \mathbf{x} = t \begin{bmatrix} -2\\ 1 \end{bmatrix}, \ t \in \mathbf{R} \right\},$ $E(7) = \left\{ \mathbf{x} = t \begin{bmatrix} 1\\ 3 \end{bmatrix}, \ t \in \mathbb{R} \right\}.$

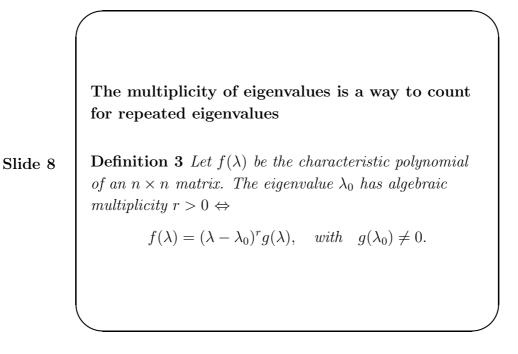
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Solving the characteristic equation one finds first the $\lambda {\rm 's}$

Theorem 2 λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$.

Slide 7 Definition 2 Given an $n \times n$ matrix A, the function $f(\lambda) = \det(A - \lambda I)$ is called the characteristic function of A.

The characteristic function of A, $n \times n$ is a polynomial in λ of degree n.



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Eigenvalue with multiplicity 2
The eigenvalues of
$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$
. are given by
 $\begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0,$
that is, $\lambda = 2$, which has multiplicity 2.

More examples of eigenvalues various multiplicities

Find the eigenvalues and eigenspaces of the following matrices:

	3	1	1			3	0	1	
A =	0	3	2	,	B =	0	3	2	.
	0	0	1 _			0	0	1	

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The both matrices have the same eigenvalues, because,

$$f_A(\lambda) = f_B(\lambda) = (\lambda - 3)^2 (1 - \lambda)$$

so the eigenvalues are:

 $\lambda=3$ with multiplicity 2; and $\lambda=1$ with multiplicity 1.

Once the eigenvalues are know, the eigenvectors can be easily computed

Slide 11 If the eigenvalues λ are known, then the eigenvector \mathbf{x}_{λ} is solution of the homogeneous equation

$$(A - \lambda I)\mathbf{x}_{\lambda} = 0.$$

Here are the eigenvectors of the previous example The eigenspaces of the matrix $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. are $E_B(3) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad E_B(1) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}.$ dim $E_B(\lambda)$ = multipl.(λ) for every eigenvalue of B, the set of all eigenvectors of B is a basis of \mathbb{R}^3 .

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Here are the eigenvectors of the previous example The eigenspaces of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ are $E_A(3) = \left\{ \begin{bmatrix} 1\\0\\0 \end{bmatrix} \right\}, \quad E_A(1) = \left\{ \begin{bmatrix} 0\\-1\\1 \end{bmatrix} \right\},$ In the case of A, where for $\lambda = 3$ holds that dim $E_A(3) <$ multipl.(3), the

set of eigenvectors of A is not a basis of \mathbb{R}^3 .

In general dim $E(\lambda) \leq$ multipl. (λ)

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