

**Math 20F.**  
**Midterm Exam 2**  
**March 8, 2006**

*Read each question carefully, and answer each question completely.*

*Show all of your work. No credit will be given for unsupported answers.*

*Write your solutions clearly and legibly. No credit will be given for illegible solutions.*

1. (9 points) Consider the linear transformations  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 + 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_2 + 3x_3 \end{bmatrix}, \quad S \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix}$$

- (a) Find a matrix  $A$  associated to  $T$  and the matrix  $B$  associated to  $S$ . Show your work.
- (b) Is  $T$  one-to-one? Is  $T$  onto? Justify your answer.
- (c) Find the matrix of the composition  $T \circ S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . Justify your answer.

- (a) Let  $A = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)]$  and  $B = [S(\mathbf{e}_1), S(\mathbf{e}_2), S(\mathbf{e}_3)]$ . Then,

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -4 \\ 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

- (b)

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -4 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 5 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \\ \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 3 \\ 0 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,  $N(A) = \{\mathbf{0}\}$ , and then  $T$  is one-to-one.

The relation  $\dim N(A) + \dim \text{Col}(A) = 3$  and  $\dim N(A) = 0$  imply that  $\dim \text{Col}(A) = 3$  and so  $T$  is surjective.

- (c) The matrix of  $T \circ S$  is  $AB$ , given by

$$AB = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 9 \\ 1 & 4 & -12 \\ 0 & 2 & 9 \end{bmatrix}.$$

2. (8 points) Let  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$  be two bases of  $\mathbb{R}^3$ , and suppose that

$$\mathbf{c}_1 = 3\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3, \quad \mathbf{c}_2 = \mathbf{b}_1 - \mathbf{b}_3, \quad \mathbf{c}_3 = \mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3.$$

- (a) Find the change of basis matrix  $P_{\mathcal{B} \leftarrow \mathcal{C}}$ . Justify your answer.  
(b) Consider the vector  $\mathbf{x} = 2\mathbf{c}_1 + \mathbf{c}_2 - 3\mathbf{c}_3$ . Find  $[\mathbf{x}]_{\mathcal{B}}$ , that is, the components of  $\mathbf{x}$  in the basis  $\mathcal{B}$ . Justify your answer.

(a)

$$[\mathbf{c}_1]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad [\mathbf{c}_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad [\mathbf{c}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = [[\mathbf{c}_1]_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}}] = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

(b)

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad [\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{C}}[\mathbf{x}]_{\mathcal{C}},$$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -2 \end{bmatrix}.$$

3. (9 points)

(a) Find the values of the constant  $k$  such that  $\det(A) = 0$ , where,

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix}.$$

(b) Determine the values of  $k$  such that the following system of equations below has more than one solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + 3x_2 + kx_3 &= 0 \\ x_1 + kx_2 + 3x_3 &= 0 \end{aligned}$$

(c) Fix  $k = 1$  and compute the coefficients  $(1, 2)$  and  $(2, 3)$  of  $A^{-1}$ . You do not need to compute the rest of the inverse matrix.

(a)

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{vmatrix}, \\ &= (1)(9 - k^2) - (1)(6 - k) + (-1)(2k - 3), \\ &= 9 - k^2 - 6 + k - 2k + 3, \\ &= -k^2 - k + 6. \end{aligned}$$

Then,  $\det(A) = 0$  implies  $k^2 + k - 6 = 0$ , that is  $k = 2$  or  $k = -3$ .

(b) The same values of  $k$  that in (a), because the matrix of coefficients of the linear system is  $A$ .

(c)  $k = 1$ , then  $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ , and  $\det(A) = -1^2 - 1 + 6 = 4$ . Also,

$$(A^{-1})_{12} = \frac{1}{\det(A)}C_{21}, \quad (A^{-1})_{23} = \frac{1}{\det(A)}C_{32}.$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = -(3 + 1) = -4,$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -(1 + 2) = -3,$$

then

$$(A^{-1})_{12} = -1, \quad (A^{-1})_{23} = -\frac{3}{4}.$$

4. (10 points) Consider the matrix  $A = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}$

- (a) Find the eigenvalues and eigenvectors of  $A$ .  
(b) Find matrices  $P$  and  $D$  such that  $A = PDP^{-1}$  where  $P$  is invertible and  $D$  diagonal.  
(c) Compute  $A^3$ .

(a)

$$0 = \begin{vmatrix} 7 - \lambda & 5 \\ 3 & -7 - \lambda \end{vmatrix} = -(7 + \lambda)(7 - \lambda) - 15 = \lambda^2 - 49 - 15 = \lambda^2 - 8^2,$$

then  $\lambda_+ = 8$ , and  $\lambda_- = -8$ . The eigenvectors are the following:

For  $\lambda_+ = 8$ ,

$$\begin{bmatrix} -1 & 5 \\ 3 & -15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 = 5x_2, \Rightarrow \mathbf{x}_+ = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

For  $\lambda_- = -8$ ,

$$\begin{bmatrix} 15 & 5 \\ 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 3x_1 = -x_2, \Rightarrow \mathbf{x}_- = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

(b)

$$P = \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}, \quad P^{-1} = \frac{1}{16} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix}.$$

Then, one gets

$$A = \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix} \frac{1}{16} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix}$$

$$A = \frac{1}{2} \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 17 & 10 \\ 6 & -14 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}.$$

(c)

$$A^3 = PD^3P^{-1} = \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 8^3 & 0 \\ 0 & -8^3 \end{bmatrix} \frac{1}{16} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix},$$

$$A = 8^2 \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix} \frac{1}{16} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix},$$

$$A^3 = 8^2 A.$$