Math 20F
Quiz 2 (version 1)
April 22, 2005

1. (1.7.14) Find the value of h for which the set of vectors

$$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ h \end{bmatrix}$$

is linearly *dependent*. Justify your answer.

$\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ h \end{bmatrix} \right\}$ will be linearly dependent if and only if the homogeneous vector equation $x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 6 \\ 11 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a nontrivial solution.

The augmented matrix of the corresponding homogeneous matrix equation is $\begin{bmatrix} 1 & -5 & 2 & 0 \\ -1 & 6 & -1 & 0 \\ -2 & 11 & h & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & -5 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3+h & 0 \end{bmatrix}$, which has has nontrivial solutions when $h = -3$. Thus, the set of vectors is linearly dependent precisely when $h = -3$. A typical dependence relation is $-7 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} -5 \\ 6 \\ 11 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

2. (1.9.3) Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points (about the origin) through $\frac{3\pi}{2}$ radians (counterclockwise).

Rotating $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by $\frac{3\pi}{2}$ radians yields $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and rotating $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by $\frac{3\pi}{2}$ radians yields $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus,

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, the standard matrix for T is $[\, T(\mathbf{e}_1)\; T(\mathbf{e}_2)\,] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

3. (2.2.24) Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each b in $\mathbb{R}^n$. Explain why A must be invertible. [*Hint:* Is A row equivalent to I_n?]

Since $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, we can solve each of the n matrix equations

$$\begin{aligned} A\mathbf{u}_1 &= \mathbf{e}_1 \\ A\mathbf{u}_2 &= \mathbf{e}_2 \\ \cdots & \quad \cdots \\ A\mathbf{u}_n &= \mathbf{e}_n \end{aligned}$$

Thus, $A\,[\,\mathbf{u}_1\; \mathbf{u}_2\; \cdots\; \mathbf{u}_n\,] = [\,A\mathbf{u}_1\; A\mathbf{u}_2\; \cdots\; A\mathbf{u}_n\,] = [\,\mathbf{e}_1\; \mathbf{e}_2\; \cdots\; \mathbf{e}_n\,] = I$. This means that the matrix $B = [\,\mathbf{u}_1\; \mathbf{u}_2\; \cdots\; \mathbf{u}_n\,]$ satisfies $AB = I$, which implies that A is invertible.