Name: ________________________________ **ID Number:** __________________

TA: ________________________________ **Section Time:** __________________

Math 20D.
Exam 2
May 19, 2008

No calculators or any other devices are allowed on this exam.
Read each question carefully. If any question is not clear, ask for clarification.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
Answer each question completely, and show all your work.

1. (a) (10 points) Find the general solution $y(t)$ to the differential equation

$$9y'' - 12y' + 4y = 0. \tag{1}$$

(b) (10 points) Find the unique solution to the initial value problem given by Eq. (1) and satisfying the initial conditions

$$y(0) = 2, \qquad y'(0) = \frac{1}{3}.$$

Solution:

Part (a): The characteristic equation for the differential equation above is:

$$9r^2 - 12r + 4 = 0 \quad \Rightarrow \quad r = \frac{1}{18}\left[12 \pm \sqrt{12^2 - (4^2)(3^2)}\right]$$
$$= \frac{12}{18} \quad \Rightarrow \quad r = \frac{2}{3}.$$

Therefore, the general solution of this differential equation is

$$\boxed{y(t) = c_1 e^{\frac{2}{3}t} + c_2 t e^{\frac{2}{3}t}}.$$

Part (b): From the general solution found in part (a) we know that $y(0) = c_1$, and the initial condition $y(0) = 2$ then implies that $c_1 = 2$.

The derivative of the general solution is

$$y'(t) = \frac{2}{3}c_1 e^{\frac{2}{3}t} + \left(\frac{2}{3}t + 1\right)c_2 e^{\frac{2}{3}t} \quad \Rightarrow \quad y'(0) = \frac{2}{3}c_1 + c_2.$$

The initial condition $y'(0) = 1/3$ implies

$$\frac{1}{3} = \frac{2}{3}2 + c_2 \quad \Rightarrow \quad c_2 = -1.$$

Therefore, the unique solution to the initial value problem given in part (b) is

$$\boxed{y(t) = 2e^{\frac{2}{3}t} - te^{\frac{2}{3}t}}.$$

2. (a) (15 points) Use the method of under-determined coefficients to find the general solution $y(t)$ to the differential equation

$$y'' - 2y' - 3y = 3te^{2t}. \tag{2}$$

(b) (10 points) Find the unique solution to the initial value problem given by Eq. (2) and satisfying the initial conditions

$$y(0) = \frac{4}{3}, \qquad y'(0) = \frac{23}{3}.$$

Solution:

Part (a): The characteristic equation is

$$r^2 - 2r - 3 = 0 \quad \Rightarrow \quad \begin{aligned} r &= \frac{1}{2}\left[2 \pm \sqrt{4+12}\right] \\ &= \frac{1}{2}(2 \pm 4) \quad \Rightarrow \quad r_1 = 3, \quad r_2 = -1. \end{aligned}$$

So the general solution to the homogeneous equation is

$$y_h(t) = c_1 e^{3t} + c_2 e^{-t}.$$

A particular solution of the inhomogeneous equation must be of the form

$$y_p(t) = (at + b)e^{2t}.$$

Then we have

$$y_p'(t) = (2at + 2b + a)e^{2t}, \quad y_p''(t) = (4at + 4b + 4a)e^{2t}.$$

Introducing this information into the differential equation we obtain the values of the coefficients a and b, as follows,

$$\begin{aligned} (4at + 4b + 4a) - 2(2at + 2b + a) - 3(at + b) &= 3t \\ -3at + (2a - 3b) &= 3t \quad \Rightarrow \quad \begin{cases} -3a = 3, \\ 2a - 3b = 0, \end{cases} \end{aligned}$$

so we conclude that $a = -1$ and $b = -2/3$. $y_p(t) = -(t + 2/3)e^{2t}$. Then, the general solution to the inhomogeneous equation is

$$\boxed{y(t) = c_1 e^{3t} + c_2 e^{-t} - \left(t + \frac{2}{3}\right)e^{2t}}.$$

Part (b): We first compute the derivative of the general solution found in part (a), and we evaluate it at $t = 0$, that is,

$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - 2\left(t + \frac{2}{3}\right)e^{2t} - e^{2t} \quad \Rightarrow \quad y'(0) = 3c_1 - c_2 - \frac{4}{3} - 1.$$

Therefore, the initial condition equations are given by

$$\left.\begin{aligned} \frac{4}{3} &= y(0) = c_1 + c_2 - \frac{2}{3}, \\ \frac{23}{3} &= y'(0) = 3c_1 - c_2 - \frac{7}{3}, \end{aligned}\right\} \quad \Rightarrow \quad c_1 = 3, \quad c_2 = -1.$$

We conclude that the solution to the initial value problem is

$$\boxed{y(t) = 3e^{3t} - e^{-t} - \left(t + \frac{2}{3}\right)e^{2t}}.$$

3. (20 points) Decide whether the set of vectors shown below is linearly dependent or independent. In the case that the set of vectors is linearly dependent, express one of them as a linear combination of the other two.

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ -10 \end{bmatrix} \right\}$$

Solution: We find for nontrivial solutions of the linear system

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} c_1 + \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} c_2 + \begin{bmatrix} -5 \\ -3 \\ -10 \end{bmatrix} c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using Gauss elimination operations we obtain,

$$\begin{bmatrix} 1 & -1 & -5 \\ 3 & 1 & -3 \\ 2 & -2 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -5 \\ 0 & 4 & 12 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore we have that $c_1 = 2c_3$, $c_2 = -3c_3$ and c_3 is free. So choosing $c_3 = 1$ we obtain:

$$\boxed{\begin{bmatrix} -5 \\ -3 \\ -10 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}}.$$

4. (a) (15 points) Find a set of fundamental solutions to the equation

$$\boldsymbol{x}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \boldsymbol{x}(t). \tag{3}$$

(b) (10 points) Graph in a phase diagram the trajectories of the fundamental solutions found in part (4a). Furthermore, do a *qualitative* sketch of the trajectories of several linear combinations of these fundamental solutions.

(c) (10 points) Find the solution to the initial value problem given by Eq. (3) and the initial condition

$$\boldsymbol{x}(0) = \begin{bmatrix} -1 \\ -21 \end{bmatrix}$$

Solution:

Part (a): We find the eigenvalues and eigenvectors of the coefficient matrix. The eigenvalues are the solutions λ of the characteristic equation:

$$0 = \begin{vmatrix} (1-\lambda) & 1 \\ 4 & (-2-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 2) - 4 = \lambda^2 + \lambda - 6 \quad \Rightarrow$$

$$\lambda = \frac{1}{2}\Big(-1 \pm \sqrt{1+24}\Big) = \frac{1}{2}(-1 \pm 5) \quad \Rightarrow \quad \begin{cases} \lambda_1 = 2, \\ \lambda_2 = -3. \end{cases}$$

The eigenvectors are found solving the following systems: First, for $\lambda_1 = 2$,

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Second, for $\lambda_2 = -3$,

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}^2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Therefore, a set of fundamental solutions is given by:

$$\boxed{\boldsymbol{x}^1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}}, \qquad \boxed{\boldsymbol{x}^2(t) = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}}.$$

Part (b): The graphs is not reproduced here, but it is similar to Figure 7.5.2 (a) on page 393 in the Boyce-DiPrima Differential Equations book, eight edition.

Part (c): The general solution to the homogeneous equation is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}.$$

The initial condition implies that

$$\boldsymbol{x}(0) = \begin{bmatrix} -1 \\ -21 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix},$$

so we solve this linear system for the constants c_1 and c_2.

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & -4 & -21 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -5 & -20 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 4 \end{array}\right]$$

So, we have obtained $c_1 = -5$, $c_2 = 4$. Hence, the solution to the initial value problem is given by

$$\boxed{\boldsymbol{x}(t) = -5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + 4 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}}.$$