Name: ________________________________ **ID Number:** __________________

TA: ________________________________ **Section Time:** __________________

Math 20D
Exam 1.
April 23, 2008

No calculators or any other devices are allowed on this exam.
Read each question carefully. If any question is not clear, ask for clarification.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
Answer each question completely, and show all your work.

1. (20 points) Find the solution $y(t)$ to the initial value problem

$$t\,y' + (1+t)\,y = 3, \qquad y(1) = 0.$$

Solution: Write the equation in the following way:

$$y' + \left(\frac{1}{t} + 1\right) y = \frac{3}{t}.$$

This is an equation of the form $y' + a(t)y = b(t)$ and the solution can be computed as follows: First find the integrating factor,

$$a(t) = 1 + \frac{1}{t} \quad \Rightarrow \quad A(t) = \int a(t)dt = t + \ln(t) \quad \Rightarrow \quad \mu(t) = e^{A(t)} = te^t.$$

Second find the solution,

$$y(t) = \frac{1}{te^t}\left[c_0 + \int (te^t)\frac{3}{t}\,dt\right] = \frac{e^{-t}}{t}(c_0 + 3e^t) = \frac{c_0}{t}\,e^{-t} + \frac{3}{t}.$$

The initial condition $y(1) = 0$ implies $0 = c_0 e^{-1} + 3$, that is $c_0 = -3e$. Therefore,

$$y(t) = -\frac{3}{t}\,e^{1-t} + \frac{3}{t} \quad \Rightarrow \quad \boxed{y(t) = \frac{3}{t}\left(1 - e^{1-t}\right)}$$

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2. (a) (15 points) Compute an ***implicit*** expression for the solution $y(t)$ to the initial value problem

$$y' = \frac{e^{2t} - e^{-2t}}{4 + 3y}, \qquad y(0) = 0.$$

(b) (5 points) Find the ***explicit*** expression for the solution found in part (2a).

Solution:

(a) The differential equation is separable, so we find the solution as follows:

$$\int (4 + 3y)dy = \int \left(e^{2t} - e^{-2t}\right) dt + c_0 \quad \Rightarrow \quad 4y + \frac{3}{2}\, y^2 = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} + c_0.$$

From the initial condition one obtains

$$0 = \frac{1}{2} + \frac{1}{2} + c_0 \quad \Rightarrow \quad c_0 = -1.$$

Therefore, the implicit expression for the solution $y(t)$ of the equation above is

$$\boxed{\frac{3}{2}\, y^2(t) + 4y(t) - \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t} + 1 = 0}$$

(b) The explicit expression of the solution above can be obtained finding the appropriate root of the equation above. Both roots are given by:

$$y_{\pm}(t) = \frac{1}{3}\Big[-4 \pm \sqrt{16 + 3(e^{2t} + e^{-2t}) - 6}\Big].$$

However, only the function $y_+(t)$ satisfies the initial condition, then the explicit expression for the solution is

$$\boxed{y(t) = \frac{1}{3}\Big[-4 + \sqrt{16 + 3(e^{2t} + e^{-2t}) - 6}\Big]}$$

3. (30 points) Find all solutions $y(x)$ of the differential equation

$$\left(\frac{5y^3}{x^2} + \frac{3}{x}\right) y' + \frac{3y}{x^2} + 5x = 0.$$

You can leave the solution $y(x)$ expressed in implicit form.

Solution: We first verify whether the equation above is exact or not:

$$\begin{aligned} N &= \frac{5y^3}{x^2} + \frac{3}{x} & &\Rightarrow & N_x &= -\frac{10y^3}{x^3} - \frac{3}{x^2} \\ M &= \frac{3y}{x^2} + 5x & &\Rightarrow & M_y &= \frac{3}{x^2}, \end{aligned}$$

therefore the equation above is not exact. We now verify whether there exists an integrating factor $\mu(x)$ that convert the equation into an exact equation:

$$\frac{\mu_x(x)}{\mu(x)} = \frac{1}{N}\big(M_y - N_x\big) = \frac{1}{\frac{5y^3}{x^2} + \frac{3}{x}}\left[\frac{3}{x^2} + \frac{10y^3}{x^3} + \frac{3}{x^2}\right] = \frac{2}{x}\left[\frac{\frac{5y^3}{x^2} + \frac{3}{x}}{\frac{5y^3}{x^3} + \frac{3}{x^2}}\right] = \frac{2}{x}.$$

Therefore, the integrating factor is the solution of the equation $\mu_x/\mu = 2/x$ which is given by $\mu(x) = x^2$. The equation to solve can be transformed into the following exact equation:

$$(5y^3 + 3x)\, y' + (3y + 5x^3) = 0 \quad \Rightarrow \quad \begin{cases} N = 5y^3 + 3x & \Rightarrow \quad N_x = 3, \\ M = 3y + 5x^3 & \Rightarrow \quad M_y = 3. \end{cases}$$

Then, the solution can be computed solving the following equations:

$$\phi_y = 5y^3 + 3x \quad \Rightarrow \quad \phi = \frac{5}{4}y^4 + 3xy + g(x),$$

$$\phi_x = 3y + g_x = M = 3y + 5x^3 \quad \Rightarrow \quad g_x = 5x^3 \quad \Rightarrow \quad g(x) = \frac{5}{4}x^4 + c_0.$$

So we find that $\phi(x, y) = \frac{5}{4}y^4 + 3xy + \frac{5}{4}x^4 + c_0$. where c_0 is a constant. Therefore, the solution $y(x)$ is given implicitly by the expression;

$$\boxed{\frac{5}{4}y^4(x) + 3xy(x) + \frac{5}{4}x^4 + c_0 = 0}$$

4. (a) (10 points) Find the general solution $y(t)$ of the differential equation:

$$y'' + 2y' - 3y = 0.$$

(b) (10 points) Find the particular solutions $y_1(t)$ and $y_2(t)$ of the differential equation given in part (4a) corresponding to the initial conditions:

$$y_1(0) = 1, \quad y_1'(0) = 0, \qquad \text{and} \qquad y_2(0) = 0, \quad y_2'(0) = 1.$$

(c) (10 points) Are the solutions $y_1(t)$ and $y_2(t)$ found in part (4b) linearly independent or linearly dependent? Justify your answer, and show your work.

Solution

(a) The characteristic equation is the following:

$$r^2+2r-3=0 \quad \Rightarrow \quad r = \frac{1}{2}\big(-2\pm\sqrt{4+12}\big) = \frac{1}{2}(-2\pm 4) \quad \Rightarrow \quad r_1 = 1, \; r_2 = -3.$$

Then, the general solution is given by:

$$\boxed{y(t) = c_1 e^t + c_2 e^{-3t}}$$

(b) Given the general solution found in the previous part we can compute two particular solutions y_1 and y_2 as follows:

$$y_1(0) = 1, \; y_1'(0) = 0 \quad \Rightarrow \quad 1 = c_1 + c_2, \; 0 = c_1 - 3c_2 \quad \Rightarrow \quad c_1 = \frac{3}{4}, \; c_2 = \frac{1}{4};$$

therefore $\boxed{y_1(t) = (3e^t + e^{-3t})/4}$ Analogously, we compute the second solution $y_2(t)$.

$$y_2(0) = 0, \; y_2'(0) = 1 \quad \Rightarrow \quad 0 = c_1 + c_2, \; 1 = c_1 - 3c_2 \quad \Rightarrow \quad c_1 = \frac{1}{4}, \; c_2 = -\frac{1}{4};$$

therefore $\boxed{y_2(t) = (e^t - e^{-3t})/4}$

(c) The solutions $y_1(t)$ and $y_2(t)$ are linearly independent since their Wronskian is different from zero in at least one point. If we choose that point to be $t = 0$, then it is clear that:

$$W_{y_1,y_2}(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

therefore, $\boxed{y_1(t), y_2(t) \text{ are l.i.}}$