**Name:** ________________________________ **ID Number:** __________________

**TA:** ________________________________ **Section Time:** __________________

**Math 20D.**
**Exam 1**
**April 23, 2008**

*No calculators or any other devices are allowed on this exam.*
*Read each question carefully. If any question is not clear, ask for clarification.*
***Write your solutions clearly and legibly; no credit will be given for illegible solutions.***
***Answer each question completely, and show all your work.***

1. (20 points) Find the solution $y(t)$ to the initial value problem

$$t\,y' + (1+t)\,y = 2, \qquad y(1) = 0.$$

***Solution:*** Write the equation in the following way:

$$y' + \left(\frac{1}{t} + 1\right) y = \frac{2}{t}.$$

This is an equation of the form $y' + a(t)y = b(t)$ and the solution can be computed as follows: First find the integrating factor,

$$a(t) = 1 + \frac{1}{t} \quad \Rightarrow \quad A(t) = \int a(t)dt = t + \ln(t) \quad \Rightarrow \quad \mu(t) = e^{A(t)} = te^t.$$

Second find the solution,

$$y(t) = \frac{1}{te^t}\left[c_0 + \int (te^t)\frac{2}{t}\,dt\right] = \frac{e^{-t}}{t}(c_0 + 2e^t) = \frac{c_0}{t}\,e^{-t} + \frac{2}{t}.$$

The initial condition $y(1) = 0$ implies $0 = c_0e^{-1} + 2$, that is $c_0 = -2e$. Therefore,

$$y(t) = -\frac{2}{t}\,e^{1-t} + \frac{2}{t} \quad \Rightarrow \quad \boxed{y(t) = \frac{2}{t}\left(1 - e^{1-t}\right)}$$

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2. (a) (15 points) Compute an ***implicit*** expression for the solution $y(t)$ to the initial value problem

$$y' = \frac{e^{-2t} - e^{2t}}{3 + 4y}, \qquad y(0) = 0.$$

(b) (5 points) Find the ***explicit*** expression for the solution found in part (2a).

***Solution:***

(a) The differential equation is separable, so we find the solution as follows:

$$\int (3 + 4y)dy = \int \left(e^{-2t} - e^{2t}\right) dt + c_0 \quad \Rightarrow \quad 3y + 2y^2 = -\frac{1}{2}e^{-2t} - \frac{1}{2}e^{2t} + c_0.$$

From the initial condition one obtains

$$0 = -\frac{1}{2} - \frac{1}{2} + c_0 \quad \Rightarrow \quad c_0 = 1.$$

Therefore, the implicit expression for the solution $y(t)$ of the equation above is

$$\boxed{2y^2(t) + 3y(t) + \frac{1}{2}e^{-2t} + \frac{1}{2}e^{2t} - 1 = 0}$$

(b) The explicit expression of the solution above can be obtained finding the appropriate root of the equation above. Both roots are given by:

$$y_{\pm}(t) = \frac{1}{4}\Big[-3 \pm \sqrt{9 - 4(e^{-2t} + e^{2t}) + 8}\Big].$$

However, only the function $y_+(t)$ satisfies the initial condition, then the explicit expression for the solution is

$$\boxed{y(t) = \frac{1}{4}\Big[-3 + \sqrt{9 - 4(e^{-2t} + e^{2t}) + 8}\Big]}$$

3. (30 points) Find all solutions $y(x)$ of the differential equation

$$\left(\frac{3y^3}{x^2} + \frac{5}{x}\right) y' + \frac{5y}{x^2} + 3x = 0.$$

You can leave the solution $y(x)$ expressed in implicit form.

***Solution:*** We first verify whether the equation above is exact or not:

$$\begin{aligned} N &= \frac{3y^3}{x^2} + \frac{5}{x} & &\Rightarrow & N_x &= -\frac{6y^3}{x^3} - \frac{5}{x^2} \\ M &= \frac{5y}{x^2} + 3x & &\Rightarrow & M_y &= \frac{5}{x^2}, \end{aligned}$$

therefore the equation above is not exact. We now verify whether there exists an integrating factor $\mu(x)$ that convert the equation into an exact equation:

$$\frac{\mu_x(x)}{\mu(x)} = \frac{1}{N}(M_y - N_x) = \frac{1}{\frac{3y^3}{x^2} + \frac{5}{x}} \left[\frac{5}{x^2} + \frac{6y^3}{x^3} + \frac{5}{x^2}\right] = \frac{2}{x} \left[\frac{\frac{3y^3}{x^2} + \frac{5}{x}}{\frac{3y^3}{x^3} + \frac{5}{x^2}}\right] = \frac{2}{x}.$$

Therefore, the integrating factor is the solution of the equation $\mu_x/\mu = 2/x$ which is given by $\mu(x) = x^2$. The equation to solve can be transformed into the following exact equation:

$$(3y^3 + 5x)\, y' + (5y + 3x^3) = 0 \quad \Rightarrow \quad \begin{cases} N = 3y^3 + 5x & \Rightarrow \quad N_x = 5, \\ M = 5y + 3x^3 & \Rightarrow \quad M_y = 5. \end{cases}$$

Then, the solution can be computed solving the following equations:

$$\phi_y = 3y^3 + 5x \quad \Rightarrow \quad \phi = \frac{3}{4}y^4 + 5xy + g(x),$$

$$\phi_x = 5y + g_x = M = 5y + 3x^3 \quad \Rightarrow \quad g_x = 3x^3 \quad \Rightarrow \quad g(x) = \frac{3}{4}x^4 + c_0.$$

So we find that $\phi(x, y) = \frac{3}{4}y^4 + 5xy + \frac{3}{4}x^4 + c_0$. where $c_0$ is a constant. Therefore, the solution $y(x)$ is given implicitly by the expression;

$$\boxed{\frac{3}{4}y^4(x) + 5xy(x) + \frac{3}{4}x^4 + c_0 = 0}$$

4. (a) (10 points) Find the general solution $y(t)$ of the differential equation:

$$y'' - 2y' - 3y = 0.$$

(b) (10 points) Find the particular solutions $y_1(t)$ and $y_2(t)$ of the differential equation given in part (4a) corresponding to the initial conditions:

$$y_1(0) = 1, \quad y_1'(0) = 0, \qquad \text{and} \qquad y_2(0) = 0, \quad y_2'(0) = 1.$$

(c) (10 points) Are the solutions $y_1(t)$ and $y_2(t)$ found in part (4b) linearly independent or linearly dependent? Justify your answer, and show your work.

***Solution***

(a) The characteristic equation is the following:

$$r^2 - 2r - 3 = 0 \quad \Rightarrow \quad r = \frac{1}{2}\big(2 \pm \sqrt{4+12}\big) = \frac{1}{2}(2 \pm 4) \quad \Rightarrow \quad r_1 = 3, \ r_2 = -1.$$

Then, the general solution is given by:

$$\boxed{y(t) = c_1 e^{3t} + c_2 e^{-t}}$$

(b) Given the general solution found in the previous part we can compute two particular solutions $y_1$ and $y_2$ as follows:

$$y_1(0) = 1, \ y_1'(0) = 0 \quad \Rightarrow \quad 1 = c_1 + c_2, \ 0 = 3c_1 - c_2 \quad \Rightarrow \quad c_1 = \frac{1}{4}, \ c_2 = \frac{3}{4};$$

therefore $\boxed{y_1(t) = (e^{3t} + 3e^{-t})/4}$ Analogously, we compute the second solution $y_2(t)$.

$$y_2(0) = 0, \ y_2'(0) = 1 \quad \Rightarrow \quad 0 = c_1 + c_2, \ 1 = 3c_1 - c_2 \quad \Rightarrow \quad c_1 = \frac{1}{4}, \ c_2 = -\frac{1}{4};$$

therefore $\boxed{y_2(t) = (e^{3t} - e^{-t})/4}$

(c) The solutions $y_1(t)$ and $y_2(t)$ are linearly independent since their Wronskian is different from zero in at least one point. If we choose that point to be $t = 0$, then it is clear that:

$$W_{y_1,y_2}(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

therefore, $\boxed{y_1(t), y_2(t) \text{ are l.i.}}$