Name: ______________________________ **Sect. Number:** _________

TA: _________________________________ **Sect. Time:** _________

Math 20D.
Quiz 5
May 30, 2008

Answer each question completely, and show your work.
If you use extra paper, write your name on each extra page,
and staple the question page and your own added pages together.

1. (a) (20 points) Find a fundamental matrix $\psi(t)$ for the homogeneous system

$$\boldsymbol{x}'(t) = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \boldsymbol{x}(t).$$

(b) (20 points) Find the solution matrix $\phi(t)$ (which satisfies $\phi(0) = I$) for the system above, and use this matrix $\phi(t)$ to find the solution of the initial value problem

$$\boldsymbol{x}'(t) = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution: Part (a): The fundamental matrix ψ is constructed with two linearly independent solutions of the equation above. These solutions can be obtained from the eigenvalues and eigenvectors of the coefficient matrix in the equation. Its eigenvalues are the numbers λ solutions of the characteristic equation

$$0 = \begin{vmatrix} (5-\lambda) & -1 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 5)(\lambda - 1) + 3 = \lambda^2 - 6\lambda + 8$$

$$\lambda = \frac{1}{2}\big[6 \pm \sqrt{36 - 32}\big] = \frac{1}{2}(6 \pm 2) \quad \Rightarrow \quad \begin{cases} \lambda_1 = 4 \\ \lambda_2 = 2. \end{cases}$$

The eigenvector for $\lambda_1 = 4$ is a solution of the system with coefficients:

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvector for $\lambda_2 = 2$ is a solution of the system with coefficients

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \to \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}^2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Therefore, the fundamental matrix is

$$\boxed{\psi(t) = \begin{bmatrix} e^{4t} & e^{2t} \\ e^{4t} & 3e^{2t} \end{bmatrix}}.$$

Part (b): The solution matrix $\phi(t)$ is given by the formula $\phi(t) = \psi(t)\psi^{-1}(0)$. Noticing that

$$\psi(0) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \Rightarrow \quad \psi^{-1}(0) = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\phi(t) = \begin{bmatrix} e^{4t} & e^{2t} \\ e^{4t} & 3e^{2t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \phi(t) = \frac{e^{2t}}{2} \begin{bmatrix} 3e^{2t} - 1 & -e^{2t+1} \\ 3e^{2t} - 3 & -e^{2t} + 3 \end{bmatrix}.$$

The solution of the initial value problem above is then $\boldsymbol{x}(t) = \phi(t)\boldsymbol{x}(0)$, that is

$$\boldsymbol{x}(t) = \frac{e^{2t}}{2} \begin{bmatrix} 3e^{2t} - 1 & -e^{2t} + 1 \\ 3e^{2t} - 3 & -e^{2t} + 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \boxed{\boldsymbol{x}(t) = \frac{e^{2t}}{2} \begin{bmatrix} e^{2t} + 1 \\ e^{2t} + 3 \end{bmatrix}}.$$

2. Consider the inhomogeneous system

$$\boldsymbol{x}'(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{1}$$

(a) (20 points) Find the general solution of Eq. (1) using the method of underdetermined coefficients.

(b) (20 points) Find the general solution of Eq. (1) using the method of variation of parameters.

(c) (20 points) Find the general solution of Eq. (1) using that matrix $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: Part (a): The general solution of the associated homogeneous problem is constructed with the eigenvalues and eigenvectors of the coefficient matrix given above. The eigenvalues λ are the solutions of the characteristic equation

$$0 = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9 \quad \Rightarrow \quad \lambda - 1 = \pm 3 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 4, \\ \lambda_2 = -2. \end{cases}$$

An eigenvector for $\lambda_1 = 4$ is a solution of the system with coefficients

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvector for $\lambda_2 = -2$ is a solution of the system with coefficients

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}^2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So, given arbitrary constants c_1 and c_2, the general solution of the homogeneous problem is

$$\boxed{\boldsymbol{x}_h(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}}.$$

Since the source term in the differential equation is not proportional to a solution of the homogeneous problem, then the guess for the particular solution of the inhomogeneous problem is;

$$\boldsymbol{x}_p(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Introducing this constant vector into the equation we get

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

The particular solution is then

$$\boxed{\boldsymbol{x}_p(t) = -\frac{1}{8}\begin{bmatrix}5\\5\end{bmatrix}}.$$

The general solution is then,

$$\boxed{\boldsymbol{x}(t) = \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t)}.$$

Part (b): The solution of the homogeneous problem was already computed in part (a). The particular solution of the inhomogeneous problem is now computed using the variation of parameters method. The fundamental matrix $\psi(t)$ and its inverse have the form

$$\psi(t) = \begin{bmatrix}e^{4t} & -e^{-2t}\\ e^{4t} & e^{-2t}\end{bmatrix} \quad \Rightarrow \quad \psi^{-1}(t) = \frac{1}{2}\begin{bmatrix}e^{-4t} & e^{-4t}\\ -e^{2t} & e^{2t}\end{bmatrix}.$$

The particular solution $\boldsymbol{x}_p(t)$ is computed with the formula

$$\boldsymbol{x}_p(t) = \psi(t)\int \psi^{-1}(s)\begin{bmatrix}1\\2\end{bmatrix} ds = \psi(t)\int \frac{1}{2}\begin{bmatrix}e^{-4s} & e^{-4s}\\ -e^{2s} & e^{2s}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} ds \quad \Rightarrow$$

$$\boldsymbol{x}_p(t) = \psi(t)\int \frac{1}{2}\begin{bmatrix}3e^{-4s}\\ e^{2s}\end{bmatrix} ds = \psi(t)\frac{1}{8}\begin{bmatrix}-3e^{-4t}\\ 2e^{2t}\end{bmatrix} \quad \Rightarrow$$

$$\boldsymbol{x}_p(t) = \frac{1}{8}\begin{bmatrix}e^{4t} & -e^{-2t}\\ e^{4t} & e^{-2t}\end{bmatrix}\begin{bmatrix}-3e^{-4t}\\ 2e^{2t}\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-3-2\\ -3+2\end{bmatrix}$$

Therefore, we recover the particular solution

$$\boxed{\boldsymbol{x}_p(t) = -\frac{1}{8}\begin{bmatrix}5\\1\end{bmatrix}}.$$

Part (c): Let us call A the matrix of coefficients in the equation above. The coefficient matrix is diagonalizable, since it was shown in part (a) that matrix A has two linearly independent eigenvectors. The diagonal decomposition of the coefficient matrix is $A = PDP^{-1}$ with

$$D = \begin{bmatrix}4 & 0\\ 0 & -2\end{bmatrix}, \quad P = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}.$$

Denote by $\boldsymbol{b} = \begin{bmatrix}1\\2\end{bmatrix}$ the source vector in the differential equation, and then compute the new source vector:

$$P^{-1}\boldsymbol{b} = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3\\1\end{bmatrix}.$$

Introduce the new unknown vector $\boldsymbol{y}(t) = P^{-1}\boldsymbol{x}(t)$, and then multiply the whole differential equation by P^{-1}, obtaining the diagonal system

$$\boldsymbol{y}'t) = D\boldsymbol{y}(t) + \frac{1}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} y_1'(t) = 4y_1(t) + \dfrac{3}{2}, \\ y_2'(t) = -2y_2(t) + \dfrac{1}{2} \end{cases}$$

The solution of these differential equations are computed as follows:

$$e^{-4t}y_1' - 4e^{-4t}y_1 = \frac{3}{2}e^{-4t} \quad \Rightarrow \quad y_1 e^{-4t} = -\frac{3}{8}e^{-4t} + c_1 \quad \Rightarrow \quad y_1(t) = -\frac{3}{8} + c_1 e^{4t}.$$

$$e^{2t}y_2' - 2e^{2t}y_2 = \frac{1}{2}e^{2t} \quad \Rightarrow \quad y_2 e^{2t} = \frac{1}{4}e^{2t} + c_2 \quad \Rightarrow \quad y_2(t) = \frac{1}{4} + c_2 e^{-2t}.$$

The general solution is then $\boldsymbol{x}(t) = P\boldsymbol{y}(t)$, that is

$$\boldsymbol{x}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} c_1 e^{4t} \\ c_2 e^{-2t} \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 3 \\ -2 \end{bmatrix} \right) \quad \Rightarrow$$

$$\boxed{\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} - \frac{1}{8}\begin{bmatrix} 5 \\ 1 \end{bmatrix}}.$$