Name: ______________________________ **Sect. Number:** _________

TA: ________________________________ **Sect. Time:** _________

Math 20D.
Quiz 4
May 9, 2008

Answer each question completely, and show your work.
If you use extra paper, write your name on each extra page,
and staple the question page and your own added pages together.

1. (30 points) Verify that the functions $y_1(t) = t$ and $y_2(t) = te^t$ are solutions to the homogeneous differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0 \qquad t > 0,$$

and then use the method of variation of parameters to obtain a particular solution to the inhomogeneous differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = t^3 e^{3t} \qquad t > 0.$$

Solution: We first verify that the function $y_1(t) = t$ is solution of the homogeneous equation above: $y_1'(t) = 1$ and $y_1''(t) = 0$, therefore

$$t^2(0) - t(t+2)(1) + (t+2)t = 0.$$

We now verify that the function $y_2(t) = te^t$ is also solution of the homogeneous equation above: $y_2'(t) = (t+1)e^t$ and $y_2''(t) = (t+2)e^t$, therefore,

$$t^2(t+2)e^t - t(t+2)(t+1)e^t + (t+2)te^t = (t+2)te^t[t - (t+1) + 1] = 0.$$

We now construct a particular solution y_p using the variation of parameters method. Rewrite the inhomogeneous equation above as:

$$y'' - \left(1 + \frac{2}{t}\right)y' + (t+2)\frac{1}{t^2}\,y = te^{3t} \qquad t > 0.$$

Then, the solution is given by the expression

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t), \quad \text{where} \quad \begin{cases} u_1(t) = -\displaystyle\int \frac{y_2(t)g(t)}{W(t)}\,dt \\ u_2(t) = \displaystyle\int \frac{y_1(t)g(t)}{W(t)}\,dt, \end{cases}$$

where

$$g(t) = te^{3t}, \qquad W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

We first compute the Wronskian function W:

$$W(t) = t(t+1)e^t - (1)te^t \quad \Rightarrow \quad W(t) = t^2 e^t.$$

Then, the function u_1 is given by

$$u_1(t) = -\int \frac{(te^t)(te^{3t})}{t^2 e^t}\, dt = -\int e^{3t} dt \quad \Rightarrow \quad u_1(t) = -\frac{e^{3t}}{3}.$$

The function u_2 is given by

$$u_2(t) = \int \frac{(t)(te^{3t})}{t^2 e^t}\, dt = \int e^{2t} dt \quad \Rightarrow \quad u_1(t) = \frac{e^{2t}}{2}.$$

The particular solution y_p is given by

$$y_p(t) = -\frac{e^{3t}}{3}\,(t) + \frac{e^{2t}}{2}\,(te^t) \quad \Rightarrow \quad \boxed{y_p(t) = \frac{1}{6}\, te^{3t}}.$$

2. (35 points) Decide whether the set of vectors shown below is linearly dependent or independent. In the case that the set of vectors is linearly dependent, express one of them as a linear combination of the other two.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix} \right\}$$

Solution: We perform Gauss Elimination Operations to find the solution to the homogeneous equation

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} c_1 + \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} c_2 + \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix} c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We then obtain:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 6 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

We therefore have $c_1 = -3c_3$, $c_2 = -2c_3$, and c_3 free, which says that the vectors above are ***linearly dependent***. Choosing $c_3 = 1$ we obtain

$$\boxed{\begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}}.$$

3. (35 points) Find all eigenvalues and eigenvectors of matrix A below. Also find all eigenvalues and eigenvectors of the matrix B below,

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: The eigenvalues of matrix A are the solutions to the equation

$$0 = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)^2(1-\lambda) \quad \Rightarrow \quad \begin{cases} \lambda_1 = 3, \\ \lambda_2 = 1. \end{cases}$$

The eigenvectors for $\lambda_1 = 3$ are computed as follows:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which implies that $v_3 = 0$ while v_1 and v_2 are free. Therefore, there are two linearly independent eigenvectors given by

$$\boxed{\lambda_1 = 3, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}.$$

The eigenvectors for $\lambda_2 = 1$ are computed as follows:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which implies that v_3 is free while $2v_1 = -v_3$ and $v_2 = -v_3$. Therefore, choosing $v_3 = 2$, the eigenvector is given by

$$\boxed{\lambda_2 = 1, \quad \boldsymbol{w} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}}.$$

The eigenvalues of matrix B are the solutions to the equation

$$0 = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)^2(1-\lambda) \quad \Rightarrow \quad \begin{cases} \lambda_1 = 3, \\ \lambda_2 = 1. \end{cases}$$

The eigenvectors for $\lambda_1 = 3$ are computed as follows:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which implies that $v_2 = 0$ and $v_3 = 0$ while v_1 is free. Therefore, the set of linearly eigenvectors consists of only one vector, which can be chosen to be:

$$\boxed{\lambda_1 = 3, \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}.$$

The eigenvectors for $\lambda_2 = 1$ are computed as follows:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies that v_3 is free while $v_1 = 0$ and $v_2 = -v_3$. Therefore, choosing $v_3 = 1$, the eigenvector is given by

$$\boxed{\lambda_2 = 1, \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}.$$