Name: ______________________________ **Sect. Number:** _________

TA: ________________________________ **Sect. Time:** _________

Math 20D.
Quiz 3
May 2, 2008

Answer each question completely, and show your work.
If you use extra paper, write your name on each extra page,
and staple the question page and your own added pages together.

1. (30 points) Find the Wronskian of any two linearly independent solutions of the equation

$$t^2y'' + 2ty' + (t^2 - 4)y = 0, \qquad t \geqslant 1.$$

(Notice: You do not *have* to find two linearly independent solutions, only their Wronskian.)

Solution: The equation can be rewritten for $t \geqslant 0$ as follows

$$y'' + \frac{2}{t}\,y' + \Big(1 - \frac{4}{t^2}\Big)\,y = 0.$$

It is then known that the equation for the Wronskian $W_{y_1y_2}$ of two arbitrary solutions y_1 and y_2 of the equation above is given by

$$W'_{y_1y_2}(t) + \frac{2}{t}W_{y_1y_2}(t) = 0.$$

The solution of this equation can be found as follows:

$$\frac{W'_{y_1y_2}}{W_{y_1y_2}} = -\frac{2}{t} \quad \Rightarrow \quad \ln\big(W_{y_1y_2}(t)\big) = -2\ln(t) + c_0 = \ln(t^{-2}) + c_0.$$

We then conclude that $\boxed{W_{y_1y_2}(t) = \frac{c_1}{t^2}}$, where c_0 is a constant, and $c_1 = e^{c_0}$.

2. (35 points)

The function $y_1(t) = t$ is a solution to the homogeneous differential equation

$$t^2 y'' - t\,y' + y = 0, \qquad t > 0. \tag{1}$$

Use the reduction order method to find a second solution y_2 of the equation (1), linearly independent to y_1.

(Recall that the reduction order method assumes that the second solution has the form $y_2(t) = v(t) y_1(t)$, and then one solves an equation for the function v', and then one integrates that function to obtain v.)

Solution: The equation can be rewritten for $t > 0$ as follows:

$$y'' - \frac{1}{t}\,y' + \frac{1}{t^2}\,y = 0. \tag{2}$$

It is simple to verify that $y_1(t) = t$ is a solution of this equation, since straight computation shows $0 - \frac{1}{t}(1) + \frac{1}{t^2}t = 0$.

We now look for a second solution y_2 of the equation above which is linearly independent to y_1. We have to try with a function of the form $y_2(t) = v(t) y_1(t)$. We then have

$$y_2 = tv \quad \Rightarrow \quad y_2' = v + tv' \quad \Rightarrow \quad y_2'' = 2v' + tv''.$$

Introduce these expression into the differential equation,

$$\begin{aligned} 0 &= (2v' + tv'') - \frac{1}{t}(v + tv') + \frac{1}{t^2}(tv) \\ &= tv'' + (2v' - v') + \frac{1}{t}(-v + v) \\ &= tv'' + v' \quad \Rightarrow \quad \frac{v''}{v'} = -\frac{1}{t}. \end{aligned}$$

We therefore conclude that

$$\ln(v') = -\ln(t) + c_0 = \ln(t^{-1}) + c_0 \quad \Rightarrow \quad v'(t) = \frac{c_1}{t}, \qquad c_1 = e^{c_0}.$$

We then find $v(t) = c_1 \ln(t) + c_2$. We therefore conclude that the second solution to Eq. (2) is given by $y_2(t) = c_1 t \ln(t) + c_2 t$. Therefore, a linearly independent solution to $y_1(t) = t$ is given by

$$\boxed{y_2(t) = t\ln(t)}.$$

3. (35 points) Use the method of undetermined coefficients to find the general solution to the equation

$$y'' + 2y' + 5y = 13\sin(3t) + 5t.$$

Solution: Let us split the source function into two functions given by $g_1(t) = 13\sin(3t)$, and $g_2(t) = 5t$. We first must compute the general solution to the homogeneous equation, to be sure how to do the guess for the particular solution to the inhomogeneous problems. The characteristic equation is

$$r^2 + 2r + 5 = 0 \quad \Rightarrow \quad \begin{cases} r_\pm = \dfrac{1}{2}\big[-2 \pm \sqrt{4-20}\big] \\ \quad\; = -1 \pm 2i. \end{cases}$$

Therefore, the solutions to the homogeneous equation are $y_1(t) = e^{-t}\sin(2t)$, $y_2(t) = e^{-t}\cos(2t)$. Neither of the source functions g_1 nor g_2 is proportional to the solutions of the homogeneous problem. So, our guess for the solution to the inhomogeneous problem for the first source function g_1 is

$$y_{p_1}(t) = k_1\sin(3t) + k_2\cos(3t).$$

A straightforward computation implies

$$y_{p_1}' = 3k_1\cos(3t) - 3k_2\sin(3t) \quad \Rightarrow \quad y_{p_1}'' = -9k_1\sin(3t) - 9k_2\cos(3t),$$

and thus,

$$\begin{aligned} 13\sin(3t) &= -9k_1\sin(3t) - 9k_2\cos(3t) + 6k_1\cos(3t) - 6k_2\sin(3t) \\ &\quad + 5k_1\sin(3t) + 5k_2\cos(3t) \\ &= (-9k_1 - 6k_2 + 5k_1)\sin(3t) + (-9k_2 + 6k_1 + 5k_2)\cos(3t), \end{aligned}$$

which implies the following equations for k_1 and k_2,

$$\left.\begin{aligned} -4k_1 - 6k_2 &= 13 \\ 6k_1 - 4k_2 &= 0 \end{aligned}\right\} \quad \Rightarrow \quad k_1 = -1, \quad k_2 = -\frac{3}{2}.$$

So, we conclude $y_{p_1}(t) = -\sin(3t) - \frac{3}{2}\cos(3t)$. The guess for the second part of the particular solution is

$$y_{p_2}(t) = c_1 t + c_2 \quad \Rightarrow \quad y_{p_2}'(t) = c_1, \quad y_{p_2}''(t) = 0.$$

A straightforward computation implies

$$\begin{aligned} 5t &= 0 + 2c_1 + 5c_1 t + 5c_2 \\ &= (5c_1)t + (2c_1 + 5c_2) \quad \Rightarrow \quad 5 = c_1, \quad 0 = 2c_1 + 5c_2, \end{aligned}$$

that is, $c_1 = 1$ and $c_2 = -2/5$, so $y_{p_2}(t) = t - 2/5$. The general solution to the differential equation above is

$$\boxed{y(t) = e^{-t}\big[a_1\cos(2t) + a_2\sin(2t)\big] - \sin(3t) - \frac{3}{2}\cos(3t) + t - \frac{2}{5}}.$$