Name: ______________________________ **Sect. Number:** _________

TA: ________________________________ **Sect. Time:** _________

Math 20D.
Quiz 2
April 18, 2008

Answer each question completely, and show your work.
If you use extra paper, write your name on each extra page,
and staple the question page and your own added pages together.

1. (30 points) Use the substitution $v = y^{-2}$ to find the solution to the initial value problem

$$t^2 y' + 2ty - 5y^3 = 0, \qquad y(1) = \frac{1}{\sqrt{2}}, \qquad t \geqslant 1.$$

Solution: Divide the equation by y^3,

$$t^2 \frac{y'}{y^3} + 2t \frac{1}{y^2} = 5,$$

then introduce the substitution $v = y^{-2}$, which implies $v' = -2y'/y^3$, into the equation above:

$$-\frac{t^2}{2} v' + 2t\, v = 5 \quad \Rightarrow \quad v' - \frac{4}{t} v = -\frac{10}{t^2}.$$

The last equation is of the form $v' + a(t)v = b(t)$ and can be integrated as follows:

$$a(t) = -\frac{4}{t} \quad \Rightarrow \quad A(t) = -4\ln(t) = \ln(t^{-4}) \quad \Rightarrow \quad \mu(t) = e^{A(t)} = t^{-4} = \frac{1}{t^4}.$$

$$v(t) = \frac{1}{\mu(t)} \Big[c_0 + \int \mu(s) b(s)\, ds \Big] = t^4 \Big[c_0 - \int \frac{1}{s^4} \frac{10}{s^2}\, ds \Big]$$

$$v(t) = c_0 t^4 - 10\, t^4 \int s^{-6} ds = c_0 t^4 - 10\, t^4 \frac{t^{-5}}{-5} = c_0 t^4 + \frac{2}{t}.$$

The initial condition $y(1) = 1/\sqrt{2}$ implies $v(1) = 2$, so the constant c_0 is given by $2 = v(1) = c_0 + 2$, so $c_0 = 0$. Then, $v(t) = 2/t$, so this implies that the solution $y(t)$ is

$$\boxed{y(t) = \sqrt{\frac{t}{2}}}.$$

2. (30 points) Sometimes a constant equilibrium solution has the property that solutions on one side of the equilibrium approach it, while solutions on the other side of the equilibrium depart from it. Such equilibria are called *semistable*. For the following equation, determine the constant equilibrium solutions, classify each as asymptotically stable, unstable or semistable, and sketch several graphs of solutions in the ty-plane corresponding to different initial conditions y_0. Sketch the correct concavity of the solution graphs in the ty-plane.

$$y' = y(y-7)^2, \qquad -\infty < y_0 < \infty$$

Solution: The function $f(y) = y(y-7)^2$ has two roots at the points $y_1 = 0$ and $y_2 = 7$. Its derivative is given by $f'(y) = (y-7)(3y-7)$, therefore it vanishes at $y_2 = 7$ and $y_3 = 7/3$. The graphs of both functions is given in Fig. 1. The sign of f determines the intervals where $y(t)$ solution of the equation $y' = f(y)$ is increasing ($f > 0$) or decreasing ($f < 0$). The concavity of $y(t)$ is given by the sign of $y'' = f(y)f'(y)$. The resulting graph of several solutions $y(t)$ is also given in Fig. 1.

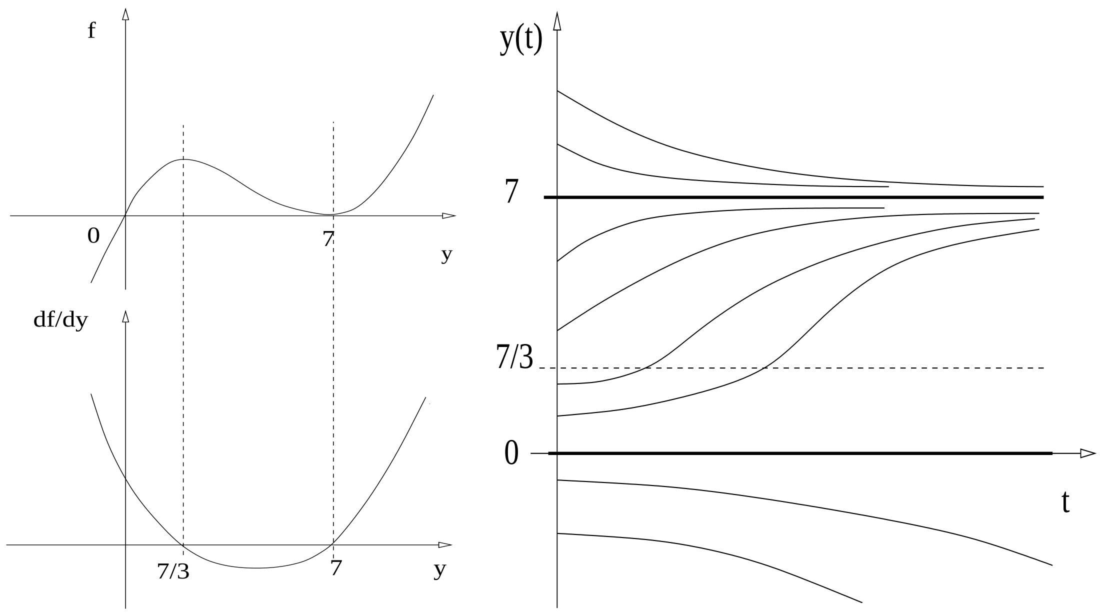

Figure 1: The graphs of functions f, f' and the solutions y for different initial conditions.

3. (40 points) Determine whether the following differential equations are exact. Only in the case that the equation is exact, find the (implicit) solution.

   (a) $\big(xe^y + x\sin(xy)\big)\, y' + xe^y + y\sin(xy) = 0.$

   (b) $(6xy + x^3)\, y' + 3y^2 + 3x^2y = 0.$

Solution:

(a)

$$\begin{aligned} N &= xe^y + x\sin(xy) \qquad &\Rightarrow \qquad N_x &= e^y + \sin(xy) + xy\cos(xy) \\ M &= xe^y + y\sin(xy) \qquad &\Rightarrow \qquad M_y &= xe^y + \sin(xy) + yx\cos(xy), \end{aligned}$$

therefore, the equation is ***not*** exact.

(b)

$$\begin{aligned} N &= 6xy + x^3 \qquad &\Rightarrow \qquad N_x &= 6y + 3x^2 \\ M &= 3y^2 + 3x^2y \qquad &\Rightarrow \qquad M_y &= 6y + 3x^2, \end{aligned}$$

therefore, the equation is exact. Then, it can be integrated as follows:

$$\phi_y = N = 6xy + x^3 \quad \Rightarrow \quad \phi = 3xy^2 + x^3y + g(x).$$

Therefore we can compute $\phi_x = 3y^2 + 3x^2y + g'(x)$. However, we also know that $\phi_x = M = 3y^2 + 3x^2y$, so we conclude that $g'(x) = 0$, and so $g(x) = c_0$, a constant. Then we have $\phi(x,y) = 3xy^2 + x^3y + c_0$, so the implicit expression for the solutions $y(x)$ is:

$$\boxed{3x\, y^2(x) + x^3 y(x) + c_0 = 0}$$