Name: ______________________________ **Sec. Number:** _________

TA: ________________________________ **Sec. Time:** _________

Math 20D.
Quiz 1
April 11, 2008

Answer each question completely, and show your work.
If you use extra paper, write your name on each extra page,
and staple the question page and your own added pages together.

1. A radioactive material desintegrates at a rate proportional to the amount currently present. Let $q(t)$ be the amount of material present at the time t, let $q'(t)$ be its time derivative and $r > 0$ be a real constant. Then, the amount of radioactive material satisfies the differential equation $q'(t) = -r\, q(t)$.

   (a) (25 points) The half-life t_h of a radioactive material is the time required for an amount of this material to decay to $1/2$ of its original value at time $t = 0$. If the half-life is $t_h = 10$ years, find the rate of change constant r.

   (b) (25 points) Find the time required for the radioactive material to decay to $1/8$ of its original value at time $t = 0$.

Answer 1a: The solution to the differential equation $q'(t) = -r\, q(t)$, with r a positive constant is gven by

$$q(t) = q_0\, e^{-rt}.$$

with q_0 the initial condition, which is unkown. From the definition of the half-live we know that

$$\frac{q_0}{2} = q_0\, e^{-rt_h} \quad \Rightarrow \quad \ln(2) = rt_h \quad \Rightarrow \quad r = \frac{ln(2)}{t_h} \quad \Rightarrow \quad \boxed{r = \frac{\ln(2)}{10}\, \frac{1}{\text{years}}}$$

Answer 1b: From part 1a we know that

$$q(t) = q_0\, e^{-\ln(2)t/t_h}.$$

We now have to find a time t_1 such that

$$\frac{q_0}{8} = q_0\, e^{-\ln(2)\, t_1/t_h} \quad \Rightarrow \quad \ln(8) = \ln(2)\frac{t_1}{t_h} \quad \Rightarrow \quad t_1 = 3t_h \quad \Rightarrow \quad \boxed{t_1 = 30\,\text{years}}$$

2. (50 points) Find the solution $y(t)$ to the initial value problem

$$y'(t) - 4y(t) = e^{3t}, \qquad y(0) = 2.$$

Answer 2: We use the integrating factor method. The solution is given by the formula:

$$y(t) = \frac{1}{\mu(t)}\Big[y_0 + \int_0^t \mu(s)b(s)ds\Big],$$

where $y_0 = 2$, $b(t) = e^{3t}$ and the function

$$\mu(t) = e^{\int_0^t (-4)ds} = e^{-4t}.$$

So the solution has the form

$$\begin{aligned} y(t) &= e^{4t}\Big[2 + \int_0^t e^{-4s}e^{3s}\, ds\Big] \\ &= e^{4t}\Big[2 + \int_0^t e^{-s}\, ds\Big] \\ &= e^{4t}\big[2 - (e^{-t} - 1)\big] \\ &= 3e^{4t} - e^{3t} \quad \Rightarrow \quad \boxed{y(t) = 3e^{4t} - e^{3t}} \end{aligned}$$