Slide 1

Change of basis

- Review: Components of a vector in a basis.
- Change of basis.
- Review: Midterm 1.

Slide 2

Any vector can be decomposed in an unique way in terms of a basis vectors

Theorem 1 *Let V be an n-dimensional vector space, and $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis of V. Then, each vector $\mathbf{v} \in V$ has a unique decomposition*

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n, \\ &= \sum_{i=1}^{n} c_i\mathbf{u}_i. \end{aligned}$$

The n scalars c_i are called components or coordinates of $\mathbf{v}$ with respect to this basis.

Proof: The set $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is basis of V, so its span is V. Therefore, there exist numbers $c_1, \cdots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n.$$

Is this decomposition unique? Suppose that it is not unique, then there exist another numbers $d_1, \cdots, d_n$ satisfying

$$\mathbf{v} = d_1\mathbf{u}_1 + \cdots + d_n\mathbf{u}_n.$$

Subtract both equations, so one obtains

$$\mathbf{0} = (c_1 - d_1)\mathbf{u}_1 + \cdots + (c_n - d_n)\mathbf{u}_n.$$

Recalling that $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a basis, and so these vectors are l.i., then each coefficient in equation above must vanish. That is, $c_1 = d_1, \cdots, c_n = d_n$. Therefore, one concludes that the decomposition of $\mathbf{v}$ is unique indeed. □

Slide 3

Gauss elimination is the tool to compute components of a vector in a basis

- Find the components of $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in the basis $\left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.
- Find the components of $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the basis $\left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

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The example above is a change of basis problem

Consider the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ given by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consider a second basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ given by

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the components of $\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2$ in the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_e, \quad [\mathbf{x}]_e = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_e.$$

The vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$ form a basis so there exists constants c_1, c_2 such that

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_u, \quad [\mathbf{x}]_u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_e.$$

Therefore,

$$[\mathbf{x}]_e = c_1[\mathbf{u}_1]_e + c_2[\mathbf{u}_2]_e.$$

That is,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_e = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_u$$

Then one has to solve the augmented matrix

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 1 & -1 & | & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ 0 & 1 & | & -\frac{1}{2} \end{bmatrix},$$

so $c_1 = 3/2$ and $c_2 = -1/2$, and then

$$[\mathbf{x}]_e = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_e, \quad [\mathbf{x}]_u = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}_u.$$

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The components of a vector change when the basis changes

Theorem 2 (Change of basis) *Let* $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ *and* $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ *be basis of* V. *Then, there exists a unique* $n \times n$ *invertible matrix* $P_{v\leftarrow u}$ *such that*

$$[\mathbf{x}]_v = P_{v\leftarrow u}[\mathbf{x}]_u,$$

for all $\mathbf{x} \in V$. *Furthermore, the matrix* $P_{v\leftarrow u}$ *has the form*

$$P_{v\leftarrow u} = \left[[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v\right],$$

and its inverse is given by

$$[P_{v\leftarrow u}]^{-1} = P_{u\leftarrow v}.$$

Proof of Theorem 2: Both sets $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ are basis of V, then there exist a unique set of numbers $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_n\}$ such that

$$\mathbf{x} = u_1\mathbf{u}_1 + \cdots + u_n\mathbf{u}_n = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_u, \quad \mathbf{x} = v_1\mathbf{v}_1 + \cdots + v_n\mathbf{v}_n = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_v.$$

Therefore,

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_v = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_u.$$

This system of equations for $(u_1, \cdots u_n)$ has a unique solution solutions for all $(v_1, \cdots v_n)$, because the $\mathbf{u}$'s and $\mathbf{v}$'s are basis. That is, $P_{v\leftarrow u} = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v]$ is invertible. □

Slide 6

The important thing in a change of basis problem is to write down the matrix $P_{v \leftarrow u}$

(2, Sec. 4.7) Let $\{\mathbf{b}_1, \mathbf{b}_2\}$, $\{\mathbf{c}_1, \mathbf{c}_2\}$ be basis of $\mathbb{R}^2$. Let $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.

- Find $[\mathbf{x}]_c$ for $[\mathbf{x}]_b = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_b$.
- Find $[\mathbf{x}]_b$ for $[\mathbf{x}]_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_c$.

Slide 7

Polynomials of degree n can be translated into column vectors in $\mathbb{R}^n$

Consider the vector space P_2.

- Find the change of coordinate matrix from the basis

$$b = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + t^2\}$$

  to the standard basis $\{1, t, t^2\}$.
- Find the b-coordinates of $\mathbf{x} = 1 - 2t$.