Print Name: ______________________ **Student Number:** ______________

Section Time: ______________

Math 20F.
Final Exam
December 7, 2005

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

(a) (4 Pts.) Find the inverse of the matrix A.

(b) (2 Pts.) Use the part (1a) to solve the system $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

(a)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & -2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 2 \end{array}\right],$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3 & 2 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 6 & 0 & 3 & 0 & 3 & 0 \\ 0 & 6 & 3 & -6 & 3 & 0 \\ 0 & 0 & 3 & -2 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & 2 & 2 \\ 0 & 6 & 0 & -4 & 2 & 2 \\ 0 & 0 & 3 & -2 & 1 & -2 \end{array}\right],$$

$$\left[\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & 2 & 2 \\ 0 & 6 & 0 & -4 & 2 & 2 \\ 0 & 0 & 6 & -4 & 2 & -4 \end{array}\right], \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}.$$

(b)

$$\mathbf{x} = A^{-1}\mathbf{x} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0+3 \\ -3+3 \\ -3-6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}.$$

So the answer is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}.$$

2. Let $T : I\!R^2 \to I\!R^3$ be a linear transformation given by

$$T(x_1, x_2) = \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + x_2 \\ x_2 \end{bmatrix}.$$

(a) (2 Pts.) Find the matrix A associated to the linear transformation T using the standard bases in $I\!R^3$ and $I\!R^2$.

(b) (2 Pts.) Is T one-to-one? Justify your answer.

(c) (2 Pts.) Is T onto? Justify your answer.

(a)

$$T(1,0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad T(0,1) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}.$$

(b) T one-to-one $\Leftrightarrow N(A) = \{\mathbf{0}\}$, and $\mathbf{x} \in N(A) \Leftrightarrow A\mathbf{x} = \mathbf{0}$,

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 7 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, $N(A) = \{\mathbf{0}\}$, then T is one-to-one.

(c)

T is not onto, because $T : I\!R^2 \to I\!R^3$, then

$$2 = \dim N(T) + \dim \mathrm{Range}(T).$$

Because $\dim N(T) = 0$, then $\dim \mathrm{Range}(T) = 2 < 3$.

3. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be two bases of $\mathbb{R}^3$, and suppose that

$$\mathbf{c}_1 = \mathbf{b}_1 - 2\mathbf{b}_2 + \mathbf{b}_3, \quad \mathbf{c}_2 = -\mathbf{b}_2 + 3\mathbf{b}_3, \quad \mathbf{c}_3 = -2\mathbf{b}_1 + \mathbf{b}_3.$$

(a) (3 Pts.) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$. Justify your answer.

(b) (3 Pts.) Consider the vector $\mathbf{x} = \mathbf{c}_1 - 2\mathbf{c}_2 + 2\mathbf{c}_3$. Find $[\mathbf{x}]_{\mathcal{B}}$, that is, the components of $\mathbf{x}$ in the basis $\mathcal{B}$. Justify your answer.

Given any $\mathbf{x} = c_1\mathbf{c}_1 + c_2\mathbf{c}_2 + c_3\mathbf{c}_3$ then,

$$[\mathbf{x}]_{\mathcal{B}} = c_1[\mathbf{c}_1]_{\mathcal{B}} + c_2[\mathbf{c}_2]_{\mathcal{B}} + c_3[\mathbf{c}_3]_{\mathcal{B}} = \big[[\mathbf{c}_1]_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}}\big][\mathbf{x}]_{\mathcal{C}},$$

then

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \big[[\mathbf{c}_1]_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}}\big] = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}.$$

(b)

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-4 \\ -2+2 \\ 1-6+2 \end{bmatrix},$$

then

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}.$$

4. Let s be a real number, and consider the system

$$\begin{aligned} sx_1 - 2sx_2 &= -1, \\ 3x_1 + 6sx_2 &= 3. \end{aligned}$$

(a) (3 Pts.) Determine the values of the parameter s for which the system above has a unique solution.

(b) (3 Pts.) For all the values of s such that the system above has a unique solution, find that solution.

(a) The system

$$\begin{bmatrix} s & -2s \\ 3 & 6s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \Leftrightarrow A\mathbf{x} = \mathbf{b},$$

has a unique solution $\Leftrightarrow \det(A) \neq 0$.

$$\det(A) = 6s^2 + 6s = 6s(s+1) \neq 0, \quad \Leftrightarrow \quad s \neq 0, \text{ or } s \neq -1.$$

(b)

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)},$$

$$A_1(\mathbf{b}) = \begin{bmatrix} -1 & -2s \\ 3 & 6s \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} s & -1 \\ 3 & 3 \end{bmatrix},$$

then

$$\det(A_1(\mathbf{b})) = -6s + 6s = 0, \quad \det(A_2(\mathbf{b})) = 3s + 3 = 3(s+1),$$

then

$$x_1 = \frac{0}{6s(s+1)} = 0, \quad x_2 = \frac{3(s+1)}{6s(s+1)} = \frac{1}{2s}.$$

so the answer is

$$\mathbf{x} = \begin{bmatrix} 0 \\ \frac{1}{2s} \end{bmatrix}.$$

5. (a) (4 Pts.) Find all the eigenvalues of the matrix,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}.$$

(b) (1 Pts.) Can you decide how many linearly independent eigenvectors has this matrix without actually computing them? Justify your answer.

$$0 = \begin{vmatrix} 3-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -2 & 2-\lambda \end{vmatrix} = (3-\lambda)[(1-\lambda)(2-\lambda)-2] - [-(2-\lambda)+2] + 2[1-(1-\lambda)],$$

$$\begin{aligned} 0 &= (3-\lambda)[\lambda^2 - 3\lambda] - \lambda + 2\lambda, \\ &= \lambda[-(\lambda-3)^2 + 1], \\ &= -\lambda[\lambda^2 - 6\lambda + 8]. \end{aligned}$$

The solutions are $\lambda = 0$ and

$$\lambda = \frac{1}{2}[6 \pm \sqrt{36-32}] = \frac{1}{2}[6 \pm 2] = 3 \pm 1,$$

that is, $\lambda = 2$, and $\lambda = 4$.

So the eigenvalues are $0, 2, 4$.

(b)

The matrix A has 3 l.i. eigenvectors, because it has 3 different eigenvalues.

6. Let V be a vector space with inner product $(\ ,\ )$, and associated norm $\|\ \|$. Let $\mathbf{x}$, $\mathbf{y} \in V$, where $\mathbf{x}$ is and eigenvector of a matrix A with eigenvalue 2, and $\mathbf{y}$ is another eigenvector with eigenvalue -3. Assume that $\|\mathbf{x}\| = 1/3$, $\|\mathbf{y}\| = 1$ and $(\mathbf{x}, \mathbf{y}) = 0$.

   (a) (3 Pts.) Compute $\|\mathbf{v}\|$ for $\mathbf{v} = 3\mathbf{x} - \mathbf{y}$.

   (b) (3 Pts.) Compute $\|A\mathbf{v}\|$ for the $\mathbf{v}$ given above.

(a)

$$\|\mathbf{v}\|^2 = (\mathbf{v}, \mathbf{v}) = (3\mathbf{x} - \mathbf{y}, 3\mathbf{x} - \mathbf{y}) = 9(\mathbf{x}, \mathbf{x}) - 6(\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{y}) = 9\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2,$$
$$\|\mathbf{v}\|^2 = 9\frac{1}{9} + 1 = 2,$$

then $\|\mathbf{v}\| = \sqrt{2}$.

(b)

$$A\mathbf{v} = A(3\mathbf{x} - \mathbf{y}) = 3A\mathbf{x} - A\mathbf{y} = 3(2\mathbf{x}) - (-3\mathbf{y}) = 3(2\mathbf{x} + \mathbf{y}),$$

then

$$\|A\mathbf{v}\|^2 = 9(2\mathbf{x} + \mathbf{y}, 2\mathbf{x} + \mathbf{y}) = 9[4(\mathbf{x}, \mathbf{x}) + 4(\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{y})] = 9[4\frac{1}{9} + 1] = 4 + 9 = 13,$$

so, $\|A\mathbf{v}\| = \sqrt{13}$.

7. Let A be a 3×3 matrix with eigenvalues 2, -1 and 3.

   (a) (2 Pts.) Find the eigenvalues of A^{-1}.

   (b) (2 Pts.) Find the determinant of A.

   (c) (2 Pts.) Find the determinant of A^{-1}.

   (d) (2 Pts.) Find the eigenvalues of $A^2 - A$.

(a) All eigenvalues are nonzero, then,

$$A\mathbf{x} = \lambda\mathbf{x} \;\Rightarrow\; \mathbf{x} = \lambda A^{-1}\mathbf{x}, \;\Rightarrow\; A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}.$$

so the eigenvalues of A^{-1} are $1/2$, -1, $1/3$.

(b)

The eigenvalues of A are all different, so A has 3 l.i. eigenvectors and the matrix P constructed with the eigenvectors as column vectors is invertible. Matrix A is diagonalizable, and $A = PDP^{-1}$ with $D = \mathrm{diag}(2, -1, 3)$.

$$\det(A) = \det(PDP^{-1}) = \det(P)\det(D)\frac{1}{\det(P)} = \det(D) = (2)(-1)(3) = -6,$$

so $\det(A) = -6$.

(c)

$$\det(A^{-1}) = \frac{1}{\det(A)} = -\frac{1}{6}.$$

(d) Let $\mathbf{x}$ be an eigenvector of A with eigenvalue λ, then,

$$(A^2 - A)\mathbf{x} = A^2\mathbf{x} - A\mathbf{x} = \lambda^2\mathbf{x} - \lambda\mathbf{x} = \lambda(\lambda - 1)\mathbf{x},$$

so the eigenvalues of $A^2 - A$ are 2, 2, 6.

8. (a) (3 Pts.) Find a basis for both the null space and the column space of A, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -4 \\ 0 & 1 & 2 \end{bmatrix}.$$

(b) (3 Pts.) Find a vector $\mathbf{v} \in I\!R^3$ such that the column space of A be equal to the orthogonal complement of $\mathbf{v}$.

(c) (3 Pts.) Find an orthogonal basis for the column space of A.

(a)

$A\mathbf{x} = \mathbf{0}$ implies

$$\begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} x_3, \quad \Rightarrow \quad N(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

(b)

A vector $\mathbf{v}$ is orthogonal to $\text{col}(A) \Leftrightarrow A^T\mathbf{v} = \mathbf{0}$. Then

$$\begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \\ 1 & -4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} v_3, \quad \Rightarrow \quad N(A^T) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Verification:

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 0.$$

(c)

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. Then,

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{-1-2}{1+4} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}.$$

So, the basis is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix} \right\}.$$

9. (3 Pts.) Find an orthonormal basis for the subspace of $I\!R^3$ spanned by the vectors

$$\left\{ \mathbf{u}_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\},$$

using the Gram-Schmidt process starting with the vector $\mathbf{u}_1$.

Let $\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$. Then,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} - \frac{-2-6-1}{4+4+1} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

So, an rthogonal basis is

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

This basis is not orhtonormal, because

$$\|\mathbf{v}_1\| = \sqrt{4+4+1} = 3, \quad \|\mathbf{v}_2\| = \sqrt{1+1} = \sqrt{2}.$$

Then, and orthonormal basis is

$$\left\{ \mathbf{w}_1 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{w}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$