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On determinants and eigenvalues

- Review: eigenvalues and eigenvectors.
- Eigenspaces.
- Characteristic equation.
- Multiplicity of eigenvalues.

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The eigenvectors of a matrix determine directions where the action of the matrix is simple

A is an $n \times n$ matrix. Then λ is an eigenvalue of A with eigenvector $\mathbf{x} \neq 0 \Leftrightarrow A\mathbf{x} = \lambda\mathbf{x}$.

Simple means $A\mathbf{x}$ is proportional to $\mathbf{x}$.

A matrix may or may not have eigenvalues and eigenvectors.

A is invertible $\Leftrightarrow \lambda = 0$ is not an eigenvalue of A.

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Examples of eigenvalues and eigenvectors

- The matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has eigenvectors and eigenvalues

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 4, \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda_2 = -2.$$

- The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has eigenvectors and eigenvalues

$$\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \lambda_1 = 0, \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \lambda_2 = 7.$$

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Eigenspaces are the subspaces spanned by the eigenvectors

Definition 1 *Let λ be an eigenvalue of A. The eigenspace $E_A(\lambda)$ is the set of all vectors $\mathbf{x}$ solutions of $A\mathbf{x} = \lambda\mathbf{x}$.*

Theorem 1 *If λ is an eigenvector of an $n \times n$ matrix A, then the set $E_A(\lambda) \subset \mathbb{R}^n$ is a subspace.*

Eigenspaces are lines, planes, or hyperplanes through the origin

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Here are the eigenspaces of the previous examples

The matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has eigenvectors $\lambda_1 = 4$ and $\lambda_2 = -2$. The corresponding eigenspaces are

$$E(4) = \left\{ \mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ t \in I\!R \right\},$$

$$E(-2) = \left\{ \mathbf{x} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ t \in I\!R \right\}.$$

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Here are the eigenspaces of the previous examples

The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has eigenvectors $\lambda_1 = 0$ and $\lambda_2 = 7$. The corresponding eigenspaces are

$$E(0) = \left\{ \mathbf{x} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ t \in I\!R \right\},$$

$$E(7) = \left\{ \mathbf{x} = t \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ t \in I\!R \right\}.$$

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Solving the characteristic equation one finds first the λ's

Theorem 2 *λ is an eigenvalue of A* $\Leftrightarrow \det(A - \lambda I) = 0$.

Definition 2 *Given an $n \times n$ matrix A, the function $f(\lambda) = \det(A - \lambda I)$ is called the characteristic function of A.*

The characteristic function of A, $n \times n$ is a polynomial in λ of degree n.

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The multiplicity of eigenvalues is a way to count for repeated eigenvalues

Definition 3 *Let $f(\lambda)$ be the characteristic polynomial of an $n \times n$ matrix. The eigenvalue λ_0 has algebraic multiplicity $r > 0 \Leftrightarrow$*

$$f(\lambda) = (\lambda - \lambda_0)^r g(\lambda), \quad \textit{with} \quad g(\lambda_0) \neq 0.$$

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Eigenvalue with multiplicity 2

The eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. are given by

$$\begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0,$$

that is, $\lambda = 2$, which has multiplicity 2.

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More examples of eigenvalues various multiplicities

Find the eigenvalues and eigenspaces of the following matrices:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The both matrices have the same eigenvalues, because,

$$f_A(\lambda) = f_B(\lambda) = (\lambda - 3)^2(1 - \lambda)$$

so the eigenvalues are:

$\lambda = 3$ with multiplicity 2; and $\lambda = 1$ with multiplicity 1.

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Once the eigenvalues are know, the eigenvectors can be easily computed

If the eigenvalues λ are known, then the eigenvector $\mathbf{x}_\lambda$ is solution of the homogeneous equation

$$(A - \lambda I)\mathbf{x}_\lambda = 0.$$

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Here are the eigenvectors of the previous example

The eigenspaces of the matrix $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. are

$$E_B(3) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad E_B(1) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

$\dim E_B(\lambda) = \text{multipl.}(\lambda)$ for every eigenvalue of B, the set of all eigenvectors of B is a basis of $\mathbb{R}^3$.

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Here are the eigenvectors of the previous example

The eigenspaces of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ are

$$E_A(3) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad E_A(1) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\},$$

In the case of A, where for $\lambda = 3$ holds that $\dim E_A(3) < \text{multipl.}(3)$, the set of eigenvectors of A is not a basis of $\mathbb{R}^3$.

In general $\dim E(\lambda) \leq$ **multipl.**(λ)