Slide 1

A matrix is a function

- Review: Column picture.
- Matrix equation $A\mathbf{x} = \mathbf{b}$.
- A matrix is a linear function.

Slide 2

The column picture is essential for linear algebra

$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 = \mathbf{b}.$$

x_1, x_2 is a solution if $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$.

Slide 3

The column picture is essential for linear algebra

$$\begin{aligned} x_1 - x_2 &= 0, \\ -x_1 + x_2 &= 2, \\ x_1 + x_2 &= 0. \end{aligned}$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

The actual computation of the solution is usually done with Gauss elimination, regardless the picture used to describe the system.

Slide 4

The column picture suggests the Matrix equation $A\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

$$A\mathbf{x} = \mathbf{b}.$$

Slide 5

The system of equations suggests how to define the product $A\mathbf{x}$.

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} 2x_1 - x_2 + x_3 \\ -x_1 + 2x_2 - x_3 \end{bmatrix}.$$

General case: The $m \times n$ matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}.$$

Slide 6

A matrix can be thought as a function

Given $\mathbf{x} \in \mathbb{R}^n$, and and $m \times n$ matrix A, then $A\mathbf{x} \in \mathbb{R}^m$. Therefore,

$$A : \mathbb{R}^n \to \mathbb{R}^m.$$

Ex: 2×3 matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$, $A : \mathbb{R}^3 \to \mathbb{R}^2$,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{R}^3, \quad A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

Slide 7

Matrices as functions have several meanings

- Reflexions: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- Rotations: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} \cos((\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
- Dilation: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Slide 8

The product $A\mathbf{x} = \mathbf{b}$ has two important properties

The definition $A\mathbf{x} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n$, where $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ is an $m \times n$ matrix and $\mathbf{x}$ is an n-vector, satisfies the properties

- $A(c\mathbf{x}) = c\,A\mathbf{x}$,
- $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.

Slide 9

Linear functions have an important role in linear algebra

Definition 1 *A function* $T : \mathbb{R}^n \to \mathbb{R}^m$ *is called linear if*

- $T(c\mathbf{x}) = c\,T(\mathbf{x})$,
- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.

for all $c \in \mathbb{R}$ *and* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$.

A matrix A is a linear function.

Slide 10

Matrices can be added and multiplied

- Review: Matrices are functions.
- Matrix operations:
  - Linear combinations.
  - Multiplication.

Slide 11

An $m \times n$ matrix A is a function $A : \mathbb{R}^n \to \mathbb{R}^m$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. A is a 3×2 matrix, so $A : \mathbb{R}^2 \to \mathbb{R}^3$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}.$$

Slide 12

Matrices are a very particular class of functions called linear functions

$T : \mathbb{R}^m \to \mathbb{R}^n$ is linear $\Leftrightarrow T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y})$.

Examples:

$T : \mathbb{R} \to \mathbb{R}$, $T(x) = mx$. (A line through the origin, $y = mx$.)

$T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{x}) = A\mathbf{x}$.

Slide 13

Like functions, matrices can be added and multiplied by numbers

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad cA = \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix}.$$

The precise way to define the product is:

$$(cA)\mathbf{x} = c(A\mathbf{x}).$$

Slide 14

The mathematical notation explained

The equation $(cA)\mathbf{x} = c(A\mathbf{x})$ is the definition of the product of a matrix by a number, because:

$$\begin{aligned}
(cA)\mathbf{x} = c(A\mathbf{x}) &= c\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = c\begin{bmatrix} -x_1 + 2x_2 \\ 2x_1 - x_2 \end{bmatrix} \\
&= \begin{bmatrix} c(-x_1 + 2x_2) \\ c(2x_1 - x_2) \end{bmatrix} = \begin{bmatrix} -cx_1 + 2cx_2 \\ 2cx_1 - cx_2 \end{bmatrix} \\
&= \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\end{aligned}$$

Slide 15

Same size matrices can be added up

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix},$$

$$A+B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2+1 & -1+1 \\ -1+3 & 2-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

The precise way to define the product is:

$$(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}.$$

Slide 16

The mathematical notation explained

The equation $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$ is the definition of the addition of two equal size matrices because:

$$\begin{aligned}
(A+B)\mathbf{x} &= \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
A\mathbf{x} + B\mathbf{x} &= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} + \begin{bmatrix} x_1 + x_2 \\ 3x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 2x_1 + x_2 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\end{aligned}$$

Slide 17

Summary: Definition of matrix addition and multiplication by a number

Let A, AB be $m \times n$ matrices with components

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}.$$

Then,

$$cA = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}, \quad A+B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mb} \end{bmatrix}$$

Slide 18

Summary: Definition of matrix addition and multiplication by a number in column notation

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n], \quad B = [\mathbf{b}_1, \cdots, \mathbf{b}_n],$$

Then,

$$\begin{aligned} cA &= [c\mathbf{a}, \cdots, c\mathbf{a}], \\ A + B &= [\mathbf{a}_1 + \mathbf{b}_1, \cdots, \mathbf{a}_n + \mathbf{b}_n]. \end{aligned}$$

Slide 19

Example of matrix multiplication

$$\begin{array}{ccccc} A & B & \to & AB \\ 2\times 3 & 3\times 3 & & 2\times 3 \end{array}$$

$$AB = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_2 \end{bmatrix}$$

$$\begin{aligned} AB &= \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_2 \end{bmatrix} \\ &= \begin{bmatrix} 2+0-1 & 2+0+1 & 0+0-1 \\ 1+0+2 & 1+1-2 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Slide 20

Matrices with appropriate size can be multiplied

Functions can be composed:

$$x \in \mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}, \quad (f \circ g)x = f(g(x)).$$

Matrices are functions. Composition of matrices is called matrix multiplication.

$$\begin{array}{cccccccccc} A & B & \to & AB & & \mathbf{x} \in \mathbb{R}^\ell & \xrightarrow{B} & \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m, \\ m\times n & n\times \ell & & m\times \ell & & \mathbf{x} \in \mathbb{R}^\ell & \to & \xrightarrow{AB} & \to & \mathbb{R}^m \end{array}$$

$$(AB)\mathbf{x} = A(B\mathbf{x}).$$

Slide 21

The mathematical notation explained

The equation $(AB)\mathbf{x} = A(B\mathbf{x})$ defines the matrix multiplication of two appropriate size matrices because:

$$\begin{aligned}
(AB)\mathbf{x} = A(B\mathbf{x}) &= A\left([\mathbf{b}_1, \cdots, \mathbf{b}_\ell]\begin{bmatrix} x_1 \\ \vdots \\ x_\ell \end{bmatrix}\right) \\
&= A(x_1\mathbf{b}_1 + \cdots + x_\ell\mathbf{b}_\ell) \\
&= x_1 A\mathbf{b}_1 + \cdots + x_\ell A\mathbf{b}_\ell \\
&= [A\mathbf{b}_1, \cdots, A\mathbf{b}_\ell]\begin{bmatrix} x_1 \\ \vdots \\ x_\ell \end{bmatrix}.
\end{aligned}$$

$$AB = [A\mathbf{b}_1, \cdots, A\mathbf{b}_\ell].$$