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Cylindrical and spherical coordinates

- Review of Polar coordinates in $\mathbb{R}^2$.
- Cylindrical coordinates in $\mathbb{R}^3$.
- Spherical coordinates in $\mathbb{R}^3$.
- Exercises.

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Polar coordinates in $\mathbb{R}^2$

Definition 1 (Polar coordinates) *Let (x, y) be Cartesian coordinates in $\mathbb{R}^2$. Then, polar coordinates (r, θ) are defined in $\mathbb{R}^2 - \{(0,0)\}$, and given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

The inverse expression is

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$

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Definitions

Definition 2 (Cylindrical coordinates) *Let (x, y, z) be Cartesian coordinates in $\mathbb{R}^3$. Then, cylindrical coordinates (r, θ, z) are defined in $\mathbb{R}^3 - \{(0, 0, z)\}$, and given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad z = z.$$

The inverse expression is

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z.$$

Computing Riemann sums in cylindrical coordinates one obtain the following formula for triple integrals,

$$\int\int\int_R f\, dV = \int\int\int_R f(x, y, z)\, r dr\, d\theta\, dz.$$

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Definitions

Definition 3 (Spherical coordinates) *Let (x, y, z) be Cartesian coordinates in $\mathbb{R}^3$. Then, spherical coordinates (r, θ, ϕ) are defined in $\mathbb{R}^3 - \{(0, 0, z)\}$, and given by*

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right).$$

The inverse expression is

$$x = r\sin(\phi)\cos(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\phi).$$

Computing Riemann sums in spherical coordinates one obtain the following formula for triple integrals,

$$\int\int\int_R f\, dV = \int\int\int_R f(x, y, z)\, r^2\sin(\phi) dr\, d\phi\, d\theta.$$

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Exercises

- Find the volume of a cylinder of radius r_0 and height h_0.
  (Answer: $V = \pi r_0^2 h_0$.)
- Find the volume of a cone of base radius r_0 and height h_0.
  (Answer: $V = \pi r_0^2 h_0/3$.)
- (Sec. 15.8, Probl. 2) Find the solid whose volume is given by

$$V = \int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r dz\, dr\, d\theta.$$

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Exercises

- Find the volume of a sphere of radius r_0.
  (Answer: $V = (4/3)\pi r_0^3$.)
- Find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.
  (Answer: $V = \pi(2 - \sqrt{2})/3$.)
- Find the volume below the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z = \sqrt{x^2 + y^2}$.
  (Answer: $V = \pi/8$.)

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Spherical coordinates

- Review: Definition of spherical coordinates.
- Exercises.

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Definitions

Definition 4 (Spherical coordinates) *Let* (x, y, z) *be Cartesian coordinates in* $\mathbb{R}^3$*. Then, spherical coordinates* (r, θ, ϕ) *are defined as follows*

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right),$$

with the inverse expression given by

$$x = r\sin(\phi)\cos(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\phi).$$

where $0 < r$*,* $0 < \phi < \pi$*, and* $0 \leq \theta < 2\pi$*.*

Computing Riemann sums in spherical coordinates one obtain the following formula for triple integrals,

$$\int\int\int_R f\, dV = \int\int\int_R f(x, y, z)\, r^2 \sin(\phi) dr\, d\phi\, d\theta.$$

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Exercises

- Find the volume of a sphere of radius r_0.
  (Answer: $V = (4/3)\pi r_0^3$.)
- Find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.
  (Answer: $V = \pi(2 - \sqrt{2})/3$.)
- Find the volume below the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z = \sqrt{x^2 + y^2}$.
  (Answer: $V = \pi/8$.)

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Exercises

- (Probl. 20, Sec. 15.8) Compute the integral
  $$I = \int\int\int_R e^{\sqrt{x^2+y^2+z^2}} dV,$$
  where the region R is the portion within the sphere $x^2 + y^2 + z^2 = 9$ in the fist octant.
  (Answer: $I = \pi(5e^3 - 1)/2$.)
- (Probl. 35, Sec. 15.8) Change to spherical coordinates and compute the following integral,
  $$I = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} x\sqrt{x^2 + y^2 + z^2} dz\, dy\, dx.$$
  (Answer: $I = 3^5\pi/5$.)