Slide 1

Double integrals on regions (Sec. 15.3)

- Regions function of y.
- Regions function of x.
- Properties of double integrals.

Slide 2

Regions functions of y

Theorem 1 (Type I) *Let $g_0(x)$, $g_1(x)$ be two continuous functions defined on an interval $[x_0, x_1]$, and such that $g_0(x) \leq g_1(x)$. Let $f(x, y)$ be a continuous function in*

$$D = \{(x, y) \in \mathbb{R}^2 : x_0 \leq x \leq x_1, \quad g_0(x) \leq y \leq g_1(x)\}.$$

Then, the integral of $f(x, y)$ in D is given by

$$\int\int_D f(x, y)\, dxdy = \int_{x_0}^{x_1} \left[\int_{g_0(x)}^{g_1(x)} f(x, y) dy \right] dx.$$

Slide 3

Example: Type I

- Find the $\int\int_D f(x,y)\,dxdy$ for

$$f(x,y) = x^2 + y^2,$$

$$D = \{(x,y) \in I\!R^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}.$$

Slide 4

$$\begin{aligned}
\int\int_D f(x,y)\,dxdy &= \int_0^1 \left[\int_{x^2}^{x} (x^2+y^2)dy\right] dx, \\
&= \int_0^1 \left[x^2\,(y|_{x^2}^{x}) + \frac{1}{3}\,(y^3|_{x^2}^{x})\right] dx, \\
&= \int_0^1 \left[x^2(x-x^2) + \frac{1}{3}(x^3-x^6)\right] dx, \\
&= \int_0^1 \left[x^3 - x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^6\right] dx, \\
&= \frac{1}{4}x^4|_0^1 - \frac{1}{5}x^5|_0^1 + \frac{1}{12}x^4|_0^1 - \frac{1}{21}x^7|_0^1, \\
&= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{3\times5\times7}.
\end{aligned}$$

Slide 5

Regions functions of x

Theorem 2 (Type II) *Let $h_0(y)$, $h_1(y)$ be two continuous functions defined on an interval $[y_0, y_1]$, and such that $h_0(y) \leq h_1(y)$. Let $f(x,y)$ be a continuous function in*

$$D = \{(x,y) \in \mathbb{R}^2 : h_0(y) \leq x \leq h_1(y), \quad y_0 \leq y \leq y_1\}.$$

Then, the integral of $f(x,y)$ in D is given by

$$\int\int_D f(x,y)\,dxdy = \int_{y_0}^{y_1} \left[\int_{h_0(y)}^{h_1(y)} f(x,y)dx \right] dy.$$

Slide 6

Example type II

- Find the $\int\int_D f(x,y)\,dxdy$ for

$$f(x,y) = x^2 + y^2,$$

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}.$$

Slide 7

Notice that $h_0(y) = y$, and $h_1(y) = \sqrt{y}$. Then,

$$D = \{(x, y) \in I\!R^2 : h_0(y) = y \leq x \leq h_1(y) = \sqrt{y}, \quad y_0 \leq y \leq y_1\}.$$

$$\begin{aligned}
\int\int_D f(x, y)\, dxdy &= \int_0^1 \left[\int_y^{\sqrt{y}} (x^2 + y^2) dx \right] dy, \\
&= \int_0^1 \left[\frac{1}{3} \left(x^3|_y^{\sqrt{y}} \right) + y^2 \left(x|_y^{\sqrt{y}} \right) \right] dy, \\
&= \int_0^1 \left[\frac{1}{3}(y^{3/2} - y^3) + y^2(y^{1/2} - y) \right] dy, \\
&= \int_0^1 \left[\frac{1}{3} y^{3/2} - \frac{1}{3} y^3 + y^{5/2} - y^3 \right] dy, \\
&= \frac{1}{3}\frac{2}{5} y^{5/2}|_0^1 - \frac{1}{3}\frac{1}{4} y^4|_0^1 + \frac{2}{7} y^{7/2}|_0^1 - \frac{1}{4} y^4|_0^1, \\
&= \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{3 \times 5 \times 7}.
\end{aligned}$$

Slide 8

- Find the $\int\int_D f(x, y)\, dxdy$ for

$$f(x, y) = 1,$$

$$D = \left\{ (x, y) \in I\!R^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \right\}.$$

As type I, then,

$$g_1(x) = 3\sqrt{1 - y^2/4}, \quad g_0(x) = -3\sqrt{1 - y^2/4}.$$

As type II, then,

$$h_1(x) = 2\sqrt{1 - x^2/9}, \quad h_0(y) = -2\sqrt{1 - x^2/9}.$$

Slide 9

Integration in polar coordinates

- Review of polar coordinates.
- Riemann sums in polar coordinates.
- Double integrals in polar coordinates.
- Examples.

Slide 10

Review of polar coordinates

Definition 1 *Let (x, y) be Cartesian coordinates in $\mathbb{R}^2$. Then, polar coordinates (r, θ) are defined in $\mathbb{R}^2 - \{(0,0)\}$, and given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

The inverse expression is

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$

Notice that

$$\det\left[\frac{\partial(x,y)}{\partial(r,\theta)}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix},$$

$$\det\left[\frac{\partial(x,y)}{\partial(r,\theta)}\right] = r\left[\cos^2(\theta) + \sin^2(\theta)\right] = r.$$

Slide 11

Riemann sums in polar coordinates

Definition 2 (Integral on disk sections) *Let $f(r,\theta)$ be a function defined on a domain*

$$D = \{(r,\theta) : 0 < r_0 \le r \le \tilde{r}_0, \quad \theta_0 \le \theta \le \tilde{\theta}_0 < 2\pi\}.$$

The integral of $f(x,y)$ in D is the number given by

$$\int\int_D f(x)\,dA = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(r_i^*, \theta_j^*)\, r_i^* \Delta r \Delta\theta,$$

if the limit exists. Given a natural number n we have introduced a partition on D by angular sections of side $\Delta r = (\tilde{r}_0 - r_0)/n$, $\Delta\theta = (\tilde{\theta}_0 - \theta_0)/n$. We denoted $r_i^* = (r_i + r_{i-1})/2$, $\theta_j^* = (\theta_j + \theta_{j-1})/2$, where $r_i = r_0 + i\Delta r$, and $\theta_j = \theta_0 + j\Delta\theta$, for $i,j = 0\cdots,n$. This choice of the sample point r_i^*, θ_j^* is called midpoint rule.

Slide 12

Double integrals in polar coordinates

Theorem 3 (Integrals on disk sections)
If $f(r,\theta)$ in continuous in

$$D = \{(r,\theta) : 0 < r_0 \le r \le r_1, \quad \theta_0 \le \theta \le \theta_1 < 2\pi\},$$

then

$$\int\int_D f(r,\theta)\,dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r,\theta)\, r\, dr d\theta.$$

This integrals on disk sections in polar coordinates are analogous to integrals on rectangular sections in Cartesian coordinates. Analogous in the sense that the limits of integrations are constants.

Shortly we generalize integration in polar coordinates to arbitrary domains, also denoted as type I and type II domains.

Slide 13

Examples

- Compute the integral of $f(x,y) = x^2 + 2y^2$ in the region

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq y, \quad 0 \leq x, \quad 1 \leq x^2 + y^2 \leq 2\}.$$

Translate to polar coordinates. $x = r\cos(\theta)$, $y = r\sin(\theta)$. Then

$$f(r,\theta) = r^2 + r^2\sin^2(\theta).$$

The region D is then,

$$D = \left\{(r,\theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq \sqrt{2}\right\}.$$

$$\begin{aligned}
\int\int_D f(r,\theta)dA &= \int_0^{\pi/2}\int_1^{\sqrt{2}} r^2(1+\sin^2(\theta))r\,drd\theta, \\
&= \left[\int_0^{\pi/2}(1+\sin^2(\theta))\,d\theta\right]\left[\int_1^{\sqrt{2}} r^3 dr\right],
\end{aligned}$$

Slide 14

Examples

$$\begin{aligned}
\int\int_D f(r,\theta)dA &= \left[(\theta|_0^{\pi/2}) + \int_0^{\pi/2}\frac{1}{2}(1-\cos(2\theta))\,d\theta\right]\left[\frac{1}{4}(r^4|_1^{\sqrt{2}})\right], \\
&= \left[\frac{\pi}{2} + \frac{1}{2}(\theta|_0^{\pi/2}) - \frac{1}{4}(\sin(2\theta)|_0^{\pi/2})\right]\frac{3}{4}, \\
&= \frac{3}{4}\left[\frac{\pi}{2} + \frac{\pi}{4}\right], \\
&= \frac{9}{16}\pi.
\end{aligned}$$

Slide 15

Examples

- Integrate $f(x,y) = e^{-(x^2+y^2)}$ in the region

$$D = \{(r,\theta) \in R^2 : \quad 0 \le \theta \le \pi, \quad 0 \le r \le 2\}.$$

Notice, $f(r,\theta) = e^{-r^2}$, then,

$$\int\int_D e^{-(x^2+y^2)} dA = \int_0^\pi \int_0^2 e^{-r^2} r\, dr d\theta,$$

substitute $u = r^2$, then $du = 2r\, dr$, then

$$\begin{aligned} \int\int_D e^{-(x^2+y^2)} dA &= \frac{1}{2} \int_0^\pi \int_0^4 e^{-u} du d\theta \\ &= \frac{1}{2} \int_0^\pi (-e^{-u}|_0^4) d\theta, \\ &= \frac{\pi}{2} \left(1 - \frac{1}{e^4}\right). \end{aligned}$$

Slide 16

From Cartesian to polar

Notice that constructing the Riemann sums in Cartesian coordinates and in polar coordinates, we have shown that the following result:

Theorem 4 (Cartesian to polar change of variables) *Let $f(x,y)$ be a continuous function on a domain D, where (x,y) represent Cartesian coordinates. Let (r,θ) be polar coordinates. Then the following formula holds,*

$$\int\int_D f(x,y)\, dx dy = \int\int_D f(r\cos(\theta), r\sin(\theta)) r\, dr d\theta.$$

Slide 17

Type I in polar coordinates

Theorem 5 (Type I in polar coordinates) *Let $h_0(\theta)$, $h_1(\theta)$ be two continuous functions defined on an interval $[\theta_0, \theta_1]$, and such that $0 < h_0(\theta) \leq h_1(\theta)$. Let $f(r, \theta)$ be a continuous function in*

$$D = \{(r, \theta) \in \mathbb{R}^2 : 0 < h_0(\theta) \leq r \leq h_1(\theta), \quad \theta_0 \leq \theta \leq \theta_1\}.$$

Then, the integral of $f(r, \theta)$ in D is given by

$$\int\int_D f(r, \theta)\, dA = \int_{\theta_0}^{\theta_1} \left[\int_{h_0(\theta)}^{h_1(\theta)} f(r, \theta) r\, dr \right] d\theta.$$

Slide 18

Type II in polar coordinates

Theorem 6 (Type II in polar coordinates) *Let $g_0(r)$, $g_1(r)$ be two continuous functions defined on an interval $[r_0, r_1]$, and such that $0 < g_0(r) \leq g_1(r) < 2\pi$. Let $f(r, \theta)$ be a continuous function in*

$$D = \{(r, \theta) \in \mathbb{R}^2 : 0 < r_0 \leq r \leq r_1, \quad 0 < g_0(r) \leq \theta \leq g_1(r) < 2\pi\}.$$

Then, the integral of $f(r, \theta)$ in D is given by

$$\int\int_D f(r, \theta)\, dA = \int_{r_0}^{r_1} \left[\int_{g_0(r)}^{g_1(r)} f(r, \theta) d\theta \right] r dr.$$

Slide 19

Triple integrals

- On rectangular boxes.
- On simple domains, type I, II, and III.
- On arbitrary domains.

Slide 20

Review of Riemann sums

- Single variable functions:

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x = \int_{x_0}^{x_1} f(x)dx.$$

- Two variable functions:

$$\lim_{n\to\infty} \sum_{i=0}^{n}\sum_{j=0}^{n} f(x_i^*, y_j^*)\Delta x\Delta y = \int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)dxdy.$$

- Three variable functions:

$$\lim_{n\to\infty} \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\Delta x\Delta y\Delta z$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} f(x,yz)\,dxdydz.$$

Slide 21

Rectangular boxed domains

Theorem 7 (Rectangular boxed domain) *Let $f(x, y, z)$ be a continuous function on a rectangular boxed domain $R = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$. Then,*

$$\int\int\int_R f\, dV = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, yz)\, dxdydz.$$

Furthermore, the integral does not change when performed in different order.

Slide 22

Example

- Compute the integral of $f(x, y, z) = xyz^2$ on the domain $R = [0, 1] \times [0, 2] \times [0, 3]$, that is,

$$R = \{(x, y, z) \in I\!R^3 : \ 0 \leq x \leq 1, \ 0 \leq y \leq 2, \ 0 \leq z \leq 3\}.$$

Slide 23

Examples

(Notice the order of the integrations.)

$$\begin{aligned}\int\!\!\int\!\!\int_R f\,dV &= \int_0^1\int_0^2\int_0^3 xyz^2\,dzdydx,\\ &= \int_0^1\int_0^2 xy\frac{1}{3}\left(z^3\big|_0^3\right)dydx,\\ &= \frac{27}{3}\int_0^1\int_0^2 xy\,dydx,\\ &= 9\int_0^1 x\frac{1}{2}\left(y^2\big|_0^2\right)dx,\\ &= 18\int_0^1 x\,dx,\\ &= 9.\end{aligned}$$

Slide 24

Triple integrals on simple regions

Type I, II, II, which means arbitrary shape only on the x variable, the y variable, and the z variable, respectively. For example, consider an integral type III:

Theorem 8 (Type III simple region) *Let $g_0(x,y)$, $g_1(x,y)$ be two continuous functions defined on a domain $[x_0,x_1]\times[y_0,y_1]$, and such that $g_0(x,y)\leq g_1(x,y)$. Let $f(x,y,z)$ be a continuous function in*

$$D=\{(x,y,z)\in I\!\!R^3 : x_0\leq x\leq x_1,\ y_0\leq y\leq y_1,\ g_0(x,y)\leq z\leq g_1(x,y)\}.$$

Then, the integral of $f(x,y,z)$ in D is given by

$$\int\!\!\int\!\!\int_D f(x,y)\,dxdy=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\left[\int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\,dz\right]dydx.$$

Slide 25

Arbitrary domains

Theorem 9 (Arbitrary domains) *Let $g_0(x,y)$, $g_1(x,y)$ be two continuous functions defined on a domain $[x_0,x_1]\times[y_0,y_1]$, and such that $g_0(x,y)\le g_1(x,y)$.*

Let $h_0(x)$, $h_1(x)$ be two continuous functions defined on a domain $[x_0,x_1]$, and such that $h_0(x)\le h_1(x)$.

Let $f(x,y,z)$ be a continuous function in

$$D=\{(x,y,z)\in\mathbb{R}^3:$$

$$x_0\le x\le x_1,\;\; h_0(x)\le y\le h_1(x),\;\; g_0(x,y)\le z\le g_1(x,y)\,\}.$$

Then, the integral of $f(x,y,z)$ in D is given by

$$\int\!\!\int\!\!\int_D f\,dV=\int_{x_0}^{x_1}\left[\int_{h_0(x)}^{h_1(x)}\left(\int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\,dz\right)dy\right]dx.$$

Slide 26

Examples

- Compute the volume of the region given by $x\ge 0$, $y\ge 0$, $z\ge 0$ and $3x+6y+2z\le 6$.

Notice: $0\le z\le(6-6y-3x)/2$. Then, for $z=0$ one has that $0\le y\le 1-x/2$. Then, for $z=0$, $y=0$, one has that $0\le x\le 2$. Therefore, the volume is given by:

$$\begin{aligned} V &= \int\!\!\int\!\!\int_D dV,\\ &= \int_0^2\left[\int_0^{1-x/2}\left(\int_0^{(6-6y-3x)/2} dz\right)dy\right]dx,\\ &= 3\int_0^2\left[\int_0^{1-x/2}\left(1-y-\frac{1}{2}x\right)dy\right]dx, \end{aligned}$$

Slide 27

Examples

$$\begin{aligned} V &= 3\int_0^2 \left[\int_0^{1-x/2} \left(1 - y - \frac{1}{2}x\right) dy\right] dx, \\ &= 3\int_0^2 \left[\left(1-\frac{1}{2}x\right)\left(1-\frac{1}{2}x\right) - \frac{1}{2}\left(1-\frac{1}{2}x\right)^2\right] dx, \\ &= \frac{3}{2}\int_0^2 \left(1-\frac{1}{2}x\right)^2 dx. \end{aligned}$$

Then, substitute $u = 1 - x/2$, then $du = -dx/2$, so

$$\begin{aligned} V &= 3\int_0^1 u^2 du, \\ &= 1. \end{aligned}$$

Slide 28

Examples

- Compute the triple integral of $f(x, y, z) = z$ in the region $y^2 + z^2 \leq 9$, $x \geq 0$, $y \geq 3x$ and $z \geq 0$.

$$\begin{aligned} \int\int\int_D f\,dv &= \int_0^1 \left[\int_{3x}^3 \left(\int_0^{\sqrt{9-y^2}} z\,dz\right) dy\right] dx, \\ &= \int_0^1 \left[\int_{3x}^3 \frac{1}{2}\left(z^2\Big|_0^{\sqrt{9-y^2}}\right) dy\right] dx, \\ &= \frac{1}{2}\int_0^1 \left[\int_{3x}^3 (9-y^2) dy\right] dx, \\ &= \frac{1}{2}\int_0^1 \left[27(1-x) - \frac{1}{3}\left(y^3\Big|_{3x}^3\right)\right] dx, \end{aligned}$$

Slide 29

Examples

$$\int\int\int_D f\,dv = \frac{1}{2}\int_0^1 \left[27(1-x) - 9(1-x)^3\right] dx,$$
$$= \frac{9}{2}\int_0^1 \left[3(1-x) - (1-x)^3\right] dx.$$

Substitute $u = 1 - x$, then $du = -dx$, so,

$$\int\int\int_D f\,dv = \frac{9}{2}\int_0^1 (3u - u^3) du,$$
$$= \frac{9}{2}\left[\frac{3}{2}\left(u^2\Big|_0^1\right) - \frac{1}{4}\left(u^4\Big|_0^1\right)\right],$$
$$= \frac{9}{2}\left(\frac{3}{2} - \frac{1}{4}\right),$$
$$= \frac{45}{8}.$$