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Double integrals (Sec. 15.1 - 15.2)

- Review of the integral of single variable functions.
- Definition of double integral on rectangles.
- Average of a function.
- Double integrals general domains (Sec. 15.2).
- Examples of double integrals.

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Integral of a single variable function

Definition 1 (Integral of single variable functions) *Let $f(x)$ be a function defined on a interval $x \in [a, b]$. The integral of $f(x)$ in $[a, b]$ is the number given by*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\,\Delta x,$$

if the limit exists. Given a natural number n we have introduced a partition on $[a, b]$ given by $\Delta x = (b - a)/n$. We denoted $x_i^* = (x_i + x_{i-1})/2$, where $x_i = a + i\Delta x$, $i = 0, 1, \cdots, n$. This choice of the sample point x_i^* is called midpoint rule.

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Double integrals on rectangles

Definition 2 (Double integrals on rectangles) *Let $f(x,y)$ be a function defined on a rectangle $R = [x_0, x_1] \times [y_0, y_1]$. The integral of $f(x,y)$ in R is the number given by*

$$\int\int_R f(x)\,dxdy = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\,\Delta x \Delta y,$$

if the limit exists. Given a natural number n we have introduced a partition on R by rectangles of side $\Delta x = (x_1 - x_0)/n$, $\Delta y = (y_1 - y_0)/n$. We denoted $x_i^* = (x_i + x_{i-1})/2$, $y_j = (y_j - y_{j-1})/2$, where $x_i = x_0 + i\Delta x$, and $y_j = y_0 + j\Delta y$, for $i, j = 0 \cdots, n$. This choice of the sample point x_i^*, y_j^* is called midpoint rule.

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Double integrals on rectangles

Notice: If $f(x,y) \geq 0$, then $\int\int_R f(x,y)\,dxdy = V$ the volume above R and below the surface given by the graph of $f(x,y)$.

Read example 3, Sec. 5.1.

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Average

The average value of a single variable function $f(x)$ is a number $\overline{f}$ such that the area below the graph of f in the interval $[a, b]$ is given by:

$$A = (b - a)\overline{f}.$$

Therefore, one has the formula:

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Definition 3 (Average) *The average of a function $f(x, y)$ in the domain $R = [x_0, x_1] \times [y_0, y_1]$ is denoted by $\overline{f}$, and it is given by the expression*

$$\overline{f} = \frac{1}{A(R)} \int_R f(x, y)\, dxdy,$$

with $A(R) = (x_1 - x_0)(y_1 - y_0)$ the area of the rectangle domain R.

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Double integrals

Theorem 1 (Fubini) *If $f(x, y)$ is a continuous function in $R = [x_0, x_1] \times [y_0, y_1]$, then*

$$\begin{aligned} \int\!\!\int_R f(x, y)\, dxdy &= \int_{y_0}^{y_1} \left[\int_{x_0}^{x_1} f(x, y)\, dx \right] dy, \\ &= \int_{x_0}^{x_1} \left[\int_{y_0}^{y_1} f(x, y)\, dy \right] dx. \end{aligned}$$

Notation: One also denotes the double integral as

$$\int\!\!\int_R f(x, y)\, dxdy = \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y)\, dxdy.$$

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Examples

$$\begin{aligned}
\int_1^3 \int_0^2 (xy^2 + 2x^2y^3) dx dy &= \int_1^3 \left[\int_0^2 (xy^2 + 2x^2y^3) dx \right] dy \\
&= \int_1^3 \left[\frac{1}{2} y^2 \left(x^2\big|_0^2 \right) + \frac{2}{3} y^3 \left(x^3\big|_0^2 \right) \right], \\
&= \int_1^3 \left[2y^2 + \frac{16}{3} y^3 \right] dy, \\
&= \frac{2}{3} y^3 \Big|_1^3 + \frac{16}{12} y^4 \Big|_1^3, \\
&= \frac{2}{3} 26 + \frac{4}{3} 80.
\end{aligned}$$

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Examples

$$\begin{aligned}
\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dy dx &= \int_1^4 \left[\int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dy \right] dx, \\
&= \int_1^4 \left[x \left(\ln(y)\big|_1^2 \right) + \frac{1}{2x} \left(y^2\big|_1^2 \right) \right] dx, \\
&= \int_1^4 \left[\ln(2) x + \frac{3}{2x} \right] dx, \\
&= \ln(2) \frac{1}{2}\, x^2\big|_1^4 + \frac{3}{2} \ln(x)\big|_1^4, \\
&= \frac{15}{2} \ln(2) + \frac{3}{2} \ln(4), \\
&= \left(\frac{15}{2} + 3 \right) \ln(2).
\end{aligned}$$

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Notice:

Fubini theorem, in the case of $f(x,y) = g(x)h(y)$ says that:

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} g(x)h(y)\,dy\,dx = \left(\int_{x_0}^{x_1} g(x)\,dx\right)\left(\int_{y_0}^{y_1} h(y)\,dy\right).$$

Example:

$$\begin{aligned}\int_0^2\int_0^1 \frac{1+x^2}{1+y^2}dydx &= \left[\int_0^2 (1+x^2)dx\right]\left[\int_0^1 \frac{1}{1+y^2}dy\right],\\ &= \left(x|_0^2 + \frac{1}{3}x|_0^2\right)\left(\arctan(y)|_0^1\right),\\ &= \frac{\pi}{4}\left(2+\frac{8}{3}\right).\end{aligned}$$

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Double integrals on regions (Sec. 15.3)

- Regions function of y.
- Regions function of x.
- Properties of double integrals.

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Regions functions of y

Theorem 2 (Type I) *Let $g_0(x)$, $g_1(x)$ be two continuous functions defined on an interval $[x_0, x_1]$, and such that $g_0(x) \leq g_1(x)$. Let $f(x, y)$ be a continuous function in*

$$D = \{(x, y) \in \mathbb{R}^2 : x_0 \leq x \leq x_1, \quad g_0(x) \leq y \leq g_1(x)\}.$$

Then, the integral of $f(x, y)$ in D is given by

$$\int\int_D f(x, y)\, dxdy = \int_{x_0}^{x_1} \left[\int_{g_0(x)}^{g_1(x)} f(x, y) dy \right] dx.$$

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Example: Type I

- Find the $\int\int_D f(x, y)\, dxdy$ for

$$f(x, y) = x^2 + y^2,$$

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}.$$

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$$\begin{aligned}
\int\int_D f(x,y)\,dxdy &= \int_0^1 \left[\int_{x^2}^x (x^2+y^2)dy\right] dx,\\
&= \int_0^1 \left[x^2\left(y|_{x^2}^x\right) + \frac{1}{3}\left(y^3|_{x^2}^x\right)\right] dx,\\
&= \int_0^1 \left[x^2(x-x^2) + \frac{1}{3}(x^3-x^6)\right] dx,\\
&= \int_0^1 \left[x^3 - x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^6\right] dx,\\
&= \frac{1}{4}x^4|_0^1 - \frac{1}{5}x^5|_0^1 + \frac{1}{12}x^4|_0^1 - \frac{1}{21}x^7|_0^1,\\
&= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{3\times 5\times 7}.
\end{aligned}$$

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Regions functions of x

Theorem 3 (Type II) *Let $h_0(y)$, $h_1(y)$ be two continuous functions defined on an interval $[y_0, y_1]$, and such that $h_0(y) \leq h_1(y)$. Let $f(x,y)$ be a continuous function in*

$$D = \{(x,y) \in I\!R^2 : h_0(y) \leq x \leq h_1(y), \quad y_0 \leq y \leq y_1\}.$$

Then, the integral of $f(x,y)$ in D is given by

$$\int\int_D f(x,y)\,dxdy = \int_{y_0}^{y_1} \left[\int_{h_0(y)}^{h_1(y)} f(x,y)dx\right] dy.$$

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Example type II

- Find the $\int\int_D f(x,y)\,dxdy$ for

$$f(x,y) = x^2 + y^2,$$

$$D = \{(x,y) \in I\!R^2 : 0 \le x \le 1, \quad x^2 \le y \le x\}.$$

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Notice that $h_0(y) = y$, and $h_1(y) = \sqrt{y}$. Then,

$$D = \{(x,y) \in I\!R^2 : h_0(y) = y \le x \le h_1(y) = \sqrt{y}, \quad y_0 \le y \le y_1\}.$$

$$\begin{aligned}
\int\int_D f(x,y)\,dxdy &= \int_0^1 \left[\int_y^{\sqrt{y}} (x^2+y^2)dx\right] dy, \\
&= \int_0^1 \left[\frac{1}{3}\left(x^3|_y^{\sqrt{y}}\right) + y^2\left(x|_y^{\sqrt{y}}\right)\right] dy, \\
&= \int_0^1 \left[\frac{1}{3}(y^{3/2} - y^3) + y^2(y^{1/2} - y)\right] dy, \\
&= \int_0^1 \left[\frac{1}{3}y^{3/2} - \frac{1}{3}y^3 + y^{5/2} - y^3\right] dy, \\
&= \frac{1}{3}\frac{2}{5}y^{5/2}|_0^1 - \frac{1}{3}\frac{1}{4}y^4|_0^1 + \frac{2}{7}y^{7/2}|_0^1 - \frac{1}{4}y^4|_0^1, \\
&= \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{3\times 5\times 7}.
\end{aligned}$$