Slide 1

Directional derivative and gradient vector (Sec. 14.6)

- Definition of directional derivative.
- Directional derivative and partial derivatives.
- Gradient vector.
- Geometrical meaning of the gradient.

Slide 2

Directional derivative

Definition 1 (Directional derivative) *The directional derivative of the function $f(x, y)$ at the point (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y \rangle$ if*

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{t \to 0} \frac{1}{t} \left[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0) \right],$$

if the limit exists.

Particular cases:

- $\mathbf{u} = \langle 1, 0 \rangle = \mathbf{i}$, then $D_{\mathbf{i}} f(x_0, y_0) = f_x(x_0, y_0)$.
- $\mathbf{u} = \langle 0, 1 \rangle = \mathbf{j}$, then $D_{\mathbf{j}} f(x_0, y_0) = f_y(x_0, y_0)$.

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Directional derivative

Notice: $\mathbf{u}$ unitary implies that t is the distance between the points $(x, y) = (x_0 + u_x t, y_0 + u_y t)$ and (x_0, y_0).

$$\begin{aligned} d &= |\langle x - x_0, y - y_0 \rangle|, \\ &= |\langle u_x t, u_y t \rangle|, \\ &= |t|\,|\mathbf{u}|, \\ &= |t|. \end{aligned}$$

The directional derivative of $f(x, y)$ at (x_0, y_0) along $\mathbf{u}$ is the pointwise rate of change of f with respect to the distance along the line parallel to $\mathbf{u}$ passing through (x_0, y_0).

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Directional derivative

Theorem 1 *If $f(x, y)$ is differentiable and $\mathbf{u} = \langle u_x, u_y \rangle$ is a unit vector, then*

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y.$$

Proof: Chain rule case 1, for $x(t) = x_0 + u_x t$, $y(t) = y_0 + u_y t$. Then, $z(t) = f(x(t), y(t))$.

On the one hand,

$$\begin{aligned} \left.\frac{dz}{dt}\right|_{t=0} &= \lim_{t \to 0} \frac{1}{t}[z(t) - z(0)], \\ &= \lim_{t \to 0} \frac{1}{t}[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)], \\ &= D_{\mathbf{u}} f(x_0, y_0). \end{aligned}$$

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Directional derivative

Proof: (Cont.) On the other hand,

$$\begin{aligned}\frac{dz}{dt} &= f_x(x(t), y(t))\frac{dx}{dt}(t) + f_y(x(t), y(y))\frac{dy}{dt}(t),\\ &= f_x(x(t), y(t))u_x + f_y(x(t), y(t))u_y,\end{aligned}$$

then,

$$\left.\frac{dz}{dt}\right|_{t=0} = f_x(x_0, y_0)u_x + f_y(x_0, y_0)u_y.$$

Therefore,

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)u_x + f_y(x_0, y_0)u_y.$$

□

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Directional derivative

Notice that

$$D_{\mathbf{u}}f = (\nabla f)\cdot \mathbf{u},$$

with $\nabla f = \langle f_x, f_y\rangle$.

- Let $f(x, y) = \sin(x + 2y)$. Compute the directional derivative of $f(x, y)$ at $(4, -2)$ in the direction $\theta = \pi/6$.

$$\mathbf{u} = \langle cos(\theta), \sin(\theta)\rangle, \quad \mathbf{u} = \langle\sqrt{3}/2, 1/2\rangle.$$

Also

$$f_x = \cos(x + 2y), \quad f_y = 2\cos(x + 2y),$$

then

$$\begin{aligned}D_{\mathbf{u}}f(x, y) &= \cos(x + 2y)u_x + 2\cos(x + 2y)u_y,\\ D_{\mathbf{u}}f(4, -2) &= \frac{\sqrt{3}}{2} + 1.\end{aligned}$$

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Directional derivative

Definition 2 (functions of 3 variables) *The directional derivative of the function $f(x, y, z)$ at the point (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ if*

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{1}{t} \left[f(x_0 + u_x t, y_0 + u_y t, z_0 + u_z t) - f(x_0, y_0, z_0) \right]$$

if the limit exists.

Theorem 2 *If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ is a unit vector, then*

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\, u_x + f_y(x_0, y_0, z_0)\, u_y + f_z(x_0, y_0, z_0) u_z.$$

Notice: $D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u}$, with $\nabla f = \langle f_x, f_y, f_z \rangle$.

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Gradient vector (2 or 3 variables)

Definition 3 *Let $f(x, y, z)$ be a differentiable function. Then,*

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle,$$

is called the gradient of $f(x, y, z)$.

In 2 variables: $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$.

Notation: $\nabla f = f_x \mathbf{i} + f_y bj + f_z \mathbf{k}$.

Theorem 3 *Let $f(x, y, z)$ be differentiable function. Then,*

$$D_{\mathbf{u}} f(\mathbf{x}) = (\nabla f(\mathbf{x})) \cdot \mathbf{u}.$$

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Gradient vector

The gradient vector has two main properties:

- It points in the direction of the maximum increase of f, and $|\nabla f|$ is the value of the maximum increase rate.
- ∇f is normal to the level surfaces.

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Gradient vector

Theorem 4 *Let f be a differentiable function of 2 or 3 variables. Fix $P_0 \in D(f)$, and let $\mathbf{u}$ be an arbitrary unit vector.*

Then, the maximum value of $D_{\mathbf{u}} f(P_0)$ among all possible directions is $|\nabla f(P_0)|$, and it is achieved for $\mathbf{u}$ parallel to $\nabla f(P_0)$.

Proof:

$$\begin{aligned} D_{\mathbf{u}} f(P_0) &= (\nabla f(P_0)) \cdot \mathbf{u}, \\ &= |\nabla f(P_0)|\, |\mathbf{u}| \cos(\theta), \\ &= |\nabla f(P_0)| \cos(\theta). \end{aligned}$$

But $-1 \leq \cos(\theta) \leq 1$ implies

$$-|\nabla f(P_0)| \leq D_{\mathbf{u}} f(P_0) \leq |\nabla f(P_0)|.$$

And $D_{\mathbf{u}} f(P_0) = |\nabla f(P_0)|, \Leftrightarrow \theta = 0 \Leftrightarrow \mathbf{u}$ is parallel $\nabla f(P_0)$. $\square$

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Gradient vector

Theorem 5 *Let $f(x, y, z)$ be a differentiable at P_0. Then, $\nabla f(P_0)$ is orthogonal to the plane tangent to a level surface containing P_0.*

Proof: Let $\mathbf{r}(t)$ be any differentiable curve in the level surface $f(x, y, z) = k$. Assume that $\mathbf{r}(t = 0) = \vec{OP}_0$. Then,

$$
\begin{aligned}
0 &= \frac{df}{dt}, \\
&= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}, \\
&= [\nabla f(\mathbf{r}(t))] \cdot \frac{\mathbf{r}}{dt}(t).
\end{aligned}
$$

But $(d\mathbf{r})/(dt)$ is tangent to the level surface for any choice of $\mathbf{r}(t)$. Therefore

$$[\nabla f(\mathbf{r}(t = 0))] \cdot \frac{\mathbf{r}}{dt}(t = 0) = 0$$

implies that $\nabla f(P_0)$ is orthogonal to the level surface. □

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Local and absolute extrema

- Local extrema (Max., Min.).
- Exercises.
- Absolute extrema.
- Exercises.

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Local Extrema

Definition 4 (Local maximum) *A function $f(x, y)$ has a local maximum at $(a, b) \in D(f) \Leftrightarrow f(x, y) \leq f(a, b)$ for all (x, y) near (a, b).*

Definition 5 (Local minimum) *A function $f(x, y)$ has a local minimum at $(a, b) \in D(f) \Leftrightarrow f(x, y) \geq f(a, b)$ for all (x, y) near (a, b).*

Theorem 6 *Let $f(x, y)$ be differentiable at (a, b). If f has a local maximum or minimum at (a, b) then $\nabla f(a, b) = \langle 0, 0\rangle$.*

(The tangent plane to the graph of f is horizontal:
$\mathbf{n} = \langle f_x, f_y, -1\rangle = \langle 0, 0, -1\rangle$.)

The converse is not true: It could be a saddle point.

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Local extrema

Definition 6 (Stationary point) *Let $f(x, y)$ be a differentiable function at (a, b). If $\nabla f(a, b) = \langle 0, 0\rangle$, then the point (a, b) is called a stationary point of f.*

Theorem 7 (Second derivative test) *Let (a, b) be a stationary point of $f(x, y)$, that is, $\nabla f(a, b) = \mathbf{0}$. Assume that $f(x, y)$ has continuous second derivatives in a disk with center in (a, b). Introduce the quantity*

$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2 .$$

- *If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
- *If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.*
- *If $D < 0$, then $f(a, b)$ is a saddle point.*
- *If $D = 0$ the test is inconclusive.*

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Exercise

- Find the maximum volume of a closed rectangular box with a given surface area A_0.

$$V(x,y,z) = xyz, \quad A(x,y,z) = 2xy + 2xz + 2yz.$$

But $A(x,y,z) = A_0$, then

$$z = \frac{A_0 - 2xy}{2(x+y)}, \quad \Rightarrow \quad V(x,y) = \frac{A_0 xy - 2x^2y^2}{2(x+y)}.$$

Find $\nabla V(x_0, y_0) = \langle 0, 0 \rangle$.

The result is $x_0 = y_0 = z_0 = \sqrt{A_0/6}$.

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Absolute extrema

Theorem 8 (Absolute extrema) *If $f(x,y)$ is continuous in a closed and bounded set $D \subset \mathbb{R}^2$, then f has an absolute maximum and an absolute minimum in D.*

Definition 7 (Bounded and closed sets)

- *A set $D \subset \mathbb{R}^2$ is bounded if it can be contained in a disk.*
- *A point $P \in \mathbb{R}^2$ is a boundary point of a set D if every disk with center in P always contains both points in D and points not in D.*
- *A set $D \in \mathbb{R}^2$ is closed if it contains all its boundary points.*

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Absolute extrema

Suggestions to find absolute extrema of $f(x,y)$ in D, closed and bounded.

- Find every stationary point of f.
  ($\nabla f(x,y) = \mathbf{0}$. No second derivative test needed.)
- Find the extrema (max. and min.) values of f on the boundary of D.
- The biggest (smallest) of the previous steps is the absolute maximum (minimum).

Exercise: Find the absolute extrema of $f(x,y) = 4x + 6y - x^2 - y^2$, on $D = \{(x,y) \in I\!R^2, \quad 0 \leq x \leq 4, \quad 0 \leq y \leq 5\}$.

Answer:

Absolute minimum: $(4,0)$, $(0,0)$. Absolute maximum: $(2,3)$.

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Lagrange multipliers

- Example of the method.
- Lagrange multipliers method: Maximization of functions subject to constraints.
- Examples.
- Generalization to more than one constraint.

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Example

- Find the rectangle of biggest area with fixed perimeter P_0.

The usual way to solve the problem is:

$$A(x,y) = xy, \quad P_0 = P(x,y) = 2x + 2y,$$

then $y = P_0/2 - x$, and replace it in $A(x,y)$,

$$A(x) = \frac{P_0}{2}x - x^2.$$

The stationary points of this function are

$$0 = A'(x) = \frac{P_0}{2} - 2x, \Rightarrow\ x = \frac{P_0}{4}, \Rightarrow\ y = \frac{P_0}{4}.$$

So the answer is the square of side

$$x = y = \frac{P_0}{4}.$$

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Lagrange multipliers method

- Find the maximum of $A(x,y) = xy$ subject to the constraint $P(x,y) = 2x + 2y = P_0$.

One has to find the (x,y) such that

$$\nabla A(x,y) = \lambda \nabla P(x,y), \quad P(x,y) = P_0,$$

with $\lambda \neq 0$. From the first equation one has

$$\langle y, x\rangle = \lambda\langle 2, 2\rangle, \quad \Rightarrow \quad x = 2\lambda, y = 2\lambda.$$

Then the constraint $P_0 = 2x + 2Y$ implies that $P_0 = 8\lambda$, so the answer is

$$x = y = \frac{P_0}{4}.$$

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Lagrange multipliers method

Theorem 9 *The extrema values of $f(x, y)$ subject to the constraint $g(x, y) = k$ can be obtained as follows:*

- *Find all solutions (x_0, y_0) and λ of the equations*

$$\begin{aligned} \nabla f(x_0, y_0) &= \lambda \nabla g(x_0, y_0), \\ g(x_0, y_0) = &= k. \end{aligned}$$

- *Evaluate f at every solution (x_0, y_0). The largest and smallest values are respectively the maximum and minimum values of f subject to the constraint $g = k$.*

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Lagrange multipliers method

Theorem 10 *The extrema values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ can be obtained as follows:*

- *Find all solutions (x_0, y_0, z_0) and λ of the equations*

$$\begin{aligned} \nabla f(x_0, y_0, z_0) &= \lambda \nabla g(x_0, y_0, z_0), \\ g(x_0, y_0, z_0) = &= k. \end{aligned}$$

- *Evaluate f at every solution (x_0, y_0, z_0). The largest and smallest values are respectively the maximum and minimum values of f subject to the constraint $g = k$.*

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Example of Lagrange multipliers method

- Find the rectangular box of maximum volume for fixed area.

The function is $V(x, y, z) = xyz$. The constraint function is $A(x, y, z) = 2xy + 2xz + 2yz$. The constraint is $A(x, y, z) = A_0$.

Find the (x, y, z) solutions of

$$\begin{aligned} \nabla V &= \lambda \nabla A, \\ A &= A_0. \end{aligned}$$

These equations are:

$$\begin{aligned} yz &= 2\lambda(z + y), \\ xz &= 2\lambda(x + z), \\ xy &= 2\lambda(x + y), \\ 2(xy + xz + zy) &= A_0. \end{aligned}$$

The solution is $x = y = z = \sqrt{A_0/6}$.

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Example of Lagrange multipliers method

- Find the extrema values of $f(x, y) = x^2 + y^2/4$ in the circle $x^2y^2 = 1$.

Then, $f(x, y, z) = x^2 + y^2/4$, and $g(x, y) = x^2 + y^2$. The equations are:

$$\begin{aligned} \nabla f &= \lambda \nabla g, & \Rightarrow \quad \langle 2x, y/2 \rangle &= \lambda \langle 2x, 2y \rangle, \\ g &= 1, & \Rightarrow \quad x^2 + y^2 &= 1. \end{aligned}$$

Which imply

$$\begin{aligned} x &= \lambda x, & \Rightarrow \quad (1 - \lambda)x &= 0, \\ y/2 &= 2\lambda y, & \Rightarrow \quad (1/4 - \lambda)y &= 0, \\ x^2 + y^2 &= 1. \end{aligned}$$

The solutions are: $P = (0, \pm 1)$, and $P = (\pm 1, 0)$. Then:

$f(0, \pm 1) = 1/4$, absolute minimum in the circle.

$f(\pm 1, 0) = 1$, absolute maximum in the circle.

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Generalization to two constraints

Theorem 11 *The extrema values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$ can be obtained as follows:*

- *Find all solutions (x_0, y_0, z_0) and λ of the equations*

$$\begin{aligned} \nabla f(x_0, y_0, z_0) &= \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0), \\ g(x_0, y_0, z_0) = &= k_1, \\ h(x_0, y_0, z_0) = &= k_2. \end{aligned}$$

- *Evaluate f at every solution (x_0, y_0, z_0). The largest and smallest values are respectively the maximum and minimum values of f subject to the constraint $g = k_1$ and $h = k_2$.*