Slide 1

Lines and planes

- Equations of lines (Vector, parametric, and symmetric eqs.).
- Equations of planes.
- Distance from a point to a plane.

Slide 2

Equations of lines

Definition 1 *Let P_0 be a point in space, and $\mathbf{v}$ be a nonzero vector. Fix a coordinate system with origin at O, and let $\mathbf{r}_0 = \vec{OP}_0$. Then, the set of vectors*

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R},$$

is called the line through P_0 parallel to $\mathbf{v}$.

This is the vector equation of the line.

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Equations of lines

Consider the case of 3 dimensions. In components,

$$\begin{aligned} \mathbf{r}(t) &= \langle x(t), y(t), z(t) \rangle, \\ \mathbf{r}_0 &= \langle x_0, y_0, z_0 \rangle, \\ \mathbf{v} &= \langle v_x, v_y, v_z \rangle, \end{aligned}$$

then one has

$$\begin{aligned} x(t) &= x_0 + t v_x, \\ y(t) &= y_0 + t v_y, \\ z(t) &= z_0 + t v_z. \end{aligned}$$

These are called the parametric equations of the line.

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Equations of lines

Compute t in expressions above, and denote $x = x(t)$, $y = y(t)$, and $z = z(t)$. Then,

$$t = \frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}.$$

These are called the symmetric equations of the line.

Definition 2 *Two lines $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$, and $\tilde{\mathbf{r}}(t) = \tilde{\mathbf{r}}_0 + t\tilde{\mathbf{v}}$ are parallel if and only if $\mathbf{v} = a\tilde{\mathbf{v}}$, with $a \neq 0$.*

Notice that in 2 dimensions, two lines are either parallel or they intersect (or both, when they coincide). This is not true in 3 dimensions.

Two lines in 3 dimensions are called skew lines if they are neither parallel nor they intersect.

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Equations of planes

Definition 3 *Fix a point in space, P_0, and a nonzero vector $\mathbf{n}$. The set of all points P satisfying*

$$\vec{P_0 P} \cdot \mathbf{n} = 0$$

is called the plane passing through P_0 normal to $\mathbf{n}$, and we denote it as $(P_0, \mathbf{n})$.

Theorem 1 *Fix a coordinate system with origin at O. The equation of the plane passing through P_0 normal to $\mathbf{n}$ can be written as*

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0,$$

where $\mathbf{r}_0 = \vec{OP_0}$, and $\mathbf{r} = \vec{OP}$, with P in the plane.

(Proof: $\vec{P_0 P} = \mathbf{r} - \mathbf{r}_0$.)

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Equations of planes

Theorem 2 *In components, the equation of a plane $(P_0, \mathbf{n})$ has the form*

$$(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0,$$

where

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \mathbf{n} = \langle n_x, n_y, n_z \rangle, \quad \mathbf{r} = \langle x, y, z \rangle.$$

Definition 4 *Two planes are parallel if their normals are parallel.*

Theorem 3 *The angle between two non-parallel planes is the angle between their normal vectors.*

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Distance from a point to a plane

Theorem 4 *The distance d from a point P_1 to a plane $(P_0, \mathbf{n})$ is the shortest distance from P_1 to any point in the plane, and is given by the expression*

$$d = \frac{|\vec{P_0 P} \cdot \mathbf{n}|}{|\mathbf{n}|}. \tag{1}$$

(Proof: Draw a picture of the situation. Then,

$$d = |\vec{P_0 P}| \cos(\theta),$$

where θ is the angle between $\vec{P_0 P}$ and $\mathbf{n}$. Recalling that

$$\vec{P_0 P} \cdot \mathbf{n} = |\mathbf{n}|\, |\vec{P_0 P}| \cos(\theta),$$

and that the distance is a nonnegative number, one gets Eq. (1).)

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Vector valued functions and derivatives

- Definition of Vector valued functions.
  - Limits
- Derivative of vector valued functions.
  - Definition.
  - Derivative rules.
  - Definite integrals.

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Vector valued functions

Definition 5 *A vector valued function* $\mathbf{r}(t)$ *is a function*

$$\mathbf{r} : D \subset I\!R \to R \subset I\!R^n,$$

with $n \geq 2$.

The symbol $\subset$ means "subset of". The set D is called domain of $\mathbf{r}$, and R is the range of $\mathbf{r}$.

In components, $n = 3$, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Think a vector valued function as 3 usual functions. (Usual means scalar valued functions $f : I\!R \to I\!R$.)

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Vector valued functions

There is a natural association between curves in $I\!R^n$ and vector valued functions. The curve is determined by the head points of the vector valued function $\mathbf{r}(t)$.

The independent variable t is called the parameter of the curve.

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Limits

Definition 6 *Consider the function* $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$*. The limit* $\lim_{t\to t_0} \mathbf{r}(t)$ *is defined as*

$$\lim_{t\to t_0} \mathbf{r}(t) = \left\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t) \right\rangle,$$

when such limit in each component exists.

Therefore, our definition is: The limit of vector valued functions as t approaches t_0 is the limit of its components in a Cartesian coordinate system.

Definition 7 *A function* $\mathbf{r}(t)$ *is continuous at* t_0 *if*

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

Having the idea of limit, one can introduce the idea of a derivative of a vector valued function.

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Derivative

Definition 8 *The derivative of a vector valued function* $\mathbf{r} : I\!R \to I\!R^n$*, denoted a* $\mathbf{r}'(t)$*, is given by*

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

when such limit exists.

Note: The derivative of a vector valued function is a vector.

Drawing an appropriate picture one concludes that $\mathbf{r}'(t)$ is a vector tangent to the curve given by $\mathbf{r}(t)$.

If $\mathbf{r}(t)$ represents the vector position of a particle, then $\mathbf{r}'(t)$ represents the velocity vector of that particle. That is,

$$\mathbf{v}(t) = \mathbf{r}'(t).$$

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Theorem 5 *Consider the function* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$*, where* $x(t)$*,* $y(t)$*, and* $z(t)$ *are differentiable functions. Then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

(Proof:

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h},$$

$$= \lim_{h\to 0} \left\langle \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}, \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}, \lim_{h\to 0} \frac{z(t+h)-z(t)}{h} \right\rangle$$

$$= \langle x'(t), y'(t), z'(t)\rangle.$$

)

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Differentiation rules

- $[\mathbf{v}(t) + \mathbf{w}(t)]' = \mathbf{v}'(t) + \mathbf{w}'(t)$,
  (addition);
- $[c\mathbf{v}(t)]' = c\mathbf{v}'(t)$,
  (product rule for constants);
- $[f(t)\mathbf{v}(t)]' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t)$,
  (product rule for scalar functions);
- $[\mathbf{v}(t) \cdot \mathbf{w}(t)]' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t)$,
  (product rule for dot product);
- $[\mathbf{v}(t) \times \mathbf{w}(t)]' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t)$,
  (product rule for cross product);
- $[\mathbf{v}(f(t))]' = \mathbf{v}'(f(t))f'(t)$,
  (chain rule for functions).

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Higher derivatives

The m derivative of $\mathbf{r}(t)$ is denoted as $\mathbf{r}^{(m)}(t)$ and is given by the expression $\mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]'$.

Example:

$$\begin{aligned} \mathbf{r}(t) &= \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle, \\ \mathbf{r}'(t) &= \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\ \mathbf{r}^{(2)}(t) &= (\mathbf{r}'(t))' = \langle -\cos(t), -\sin(t), 2 \rangle, \\ \mathbf{r}^{(3)}(t) &= (\mathbf{r}^{(2)}(t))' = \langle \sin(t), -\cos(t), 0 \rangle. \end{aligned}$$

If $\mathbf{r}(t)$ is the vector position of a particle, then the velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t)$, and the acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}^{(2)}(t)$.

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Definite integrals

Definition 9 *The definite integral of $\mathbf{r}(t)$ form $t \in [a, b]$ is a vector whose components are the integrals of the components of $\mathbf{r}(t)$, namely,*

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle.$$

Example:

$$\begin{aligned} \int_0^\pi \langle \cos(t), \sin(t), t \rangle dt &= \left\langle \int_0^\pi \cos(t)dt, \int_0^\pi \sin(t)dt, \int_0^\pi tdt \right\rangle, \\ &= \left\langle \sin(t)|_0^\pi, -\cos(t)|_0^\pi, \left.\frac{t^2}{2}\right|_0^\pi, \right\rangle, \\ &= \left\langle 0, 2, \frac{\pi^2}{2} \right\rangle. \end{aligned}$$

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Comment

The definitions of limit and derivative of vector valued functions were introduced component by component in a fixed, given in advance, Cartesian coordinate system.

What does happen if one needs to work in other coordinate system, say, spherical coordinates, or arbitrary coordinates?

All the notions of limit and derivatives for vector valued functions can be generalized in a way that the Cartesian coordinates system need not to be introduced. When one introduce such Cartesian coordinates systems, one recovers the definitions presented here.

We do not study in our course this coordinate independent notion of limit and derivatives.

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Arc length and arc length function

- Arc length of a curve.
- Arc length function.

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Arc length of a curve

The arc length of a curve in space is a *number*. It measures the extension of the curve.

Definition 10 *The arc length of the curve associated to a vector valued function* $\mathbf{r}(t)$*, for* $t \in [a, b]$ *is the number given by*

$$\ell_{ba} = \int_a^b |\mathbf{r}'(t)| \, dt.$$

Suppose that the curve represents the path traveled by a particle in space. Then, the definition above says that the length of the curve is the integral of the speed, $|\mathbf{v}(t)|$. So we say that the length of the curve is the distance traveled by the particle.

The formula above can be obtained as a limit procedure, adding up the lengths of a polygonal line that approximates the original curve.

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Arc length of a curve

In components, one has,

$$\begin{aligned} \mathbf{r}(t) &= \langle x(t), y(t), z(t) \rangle, \\ \mathbf{r}'(t) &= \langle x'(t), y'(t), z'(t) \rangle, \\ |\mathbf{r}'(t)| &= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}, \\ \ell_{ba} &= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt. \end{aligned}$$

The arc length of a general curve could be very hard to compute.

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Arc length function

Definition 11 *Consider a vector valued function* $\mathbf{r}(t)$*. The arc length function* $\ell(t)$ *from* $t = t_0$ *is given by*

$$\ell(t) = \int_0^t |\mathbf{r}'(u)| du.$$

Note: $\ell(t)$ is a scalar function. It satisfies $\ell(t_0) = 0$.

Note: The function $\ell(t)$ represents the length up to t of the curve given by $\mathbf{r}(t)$.

Our main application: Reparametrization of a given vector valued function $\mathbf{r}(t)$ using the arc length function.

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Arc length function

Reparametrization of a curve using the arc length function:

- With $\mathbf{r}(t)$ compute $\ell(t)$, starting at some $t = t_0$.
- Invert the function $\ell(t)$ to find the function $t(\ell)$.
  Example: $\ell(t) = 3e^{t/2}$, then $t(\ell) = 2\ln(\ell/3)$.
- Compute the composition $\mathbf{r}(\ell) = \mathbf{r}(t(\ell))$.
  That is, replace t by $t(\ell)$.

The function $\mathbf{r}(\ell)$ is the reparametrization of $\mathbf{r}(t)$ using the arc length as the new parameter.