CALCULUS IN SEVERAL VARIABLES

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Contents

1. Overview

The main objects of calculus are functions and a central idea is that of limit. This idea used on appropriate functions allows to compute both pointwise rate of change of functions and areas of arbitrary regions in the plane. The former is called differential calculus and the latter integral calculus. The fundamental theorem of calculus asserts that differential and integral calculus are deeply related. When introduced appropriately, differentiation and integration are inverse operations on appropriate functions.

One way to study a certain subject is to start with a simple case and later on concentrate on more complicated situations. A first course in calculus usually focuses on single variable functions, that is, $f : \mathbb{R} \to \mathbb{R}$, with values denoted as $f(x)$. Such functions are simple to represent graphically in the plane. The graphical interpretation of both the derivative at c and the integral in $[a, b]$ of $f(x)$ are also simple to obtain. In the first case $f'(c)$ is the slope of the line tangent of the graph of $f(x)$ at c, while in the second case $\int_a^b f(x)\,dx$ is the area of the shaded region in Fig. 1.

There are several generalizations of these ideas. One possibility is to consider scalar-valued functions of two and three variables, that is, $f : \mathbb{R}^2 \to \mathbb{R}$, with values denoted as $f(x, y)$, and $f : \mathbb{R}^3 \to \mathbb{R}$, with values denoted as $f(x, y, z)$, respectively. A different possibility is to consider vector-valued functions, $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, with values denoted as $\mathbf{r}(t)$. One reason to consider such generalizations is that functions of both types appear frequently in any mathematical description of nature. The

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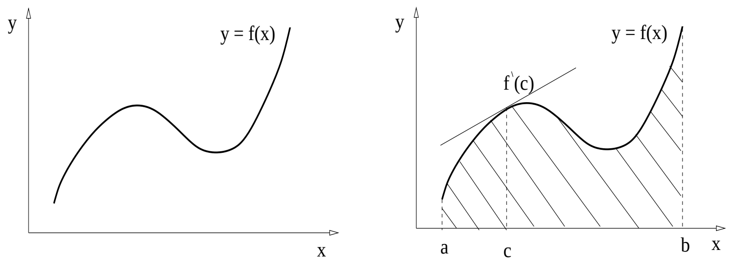


FIGURE 1. In the left picture is represented the graph of a single variable scalar function. In the right picture is represented a graphical interpretation of the derivative at a point c and the integral in the interval $[a, b]$ of that function.

vector-valued functions $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ can be represented graphically by curves is space. Hence, these functions are appropriate to describe the motion of point particles. Scalar-valued functions $(x, y) \mapsto f(x, y)$ are useful to describe situations like the temperature on the surface of a table. Here the ordered pair (x, y) label the points of the table and $f(x, y)$ its temperature. The graph of such functions can be represented graphically, as in Fig. 2 in the case of $f(x, y) = x^2 + y^2$.

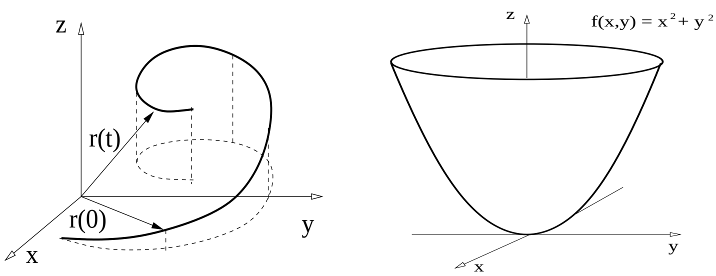


FIGURE 2. Examples of a curve in space, and the graph of a scalar function of two variables, respectively.

In vector calculus one generalizes the concept of derivative and integral to functions of several variables and to vector-valued functions. This generalization requires to introduce in space the ideas of coordinate systems, vectors, lines and planes. This presentation will occupy the first part of the course. The rest of the course will be dedicated to generalize the ideas of derivative and integral both to vector-valued functions and to scalar functions of two and three variables.

2. Vectors in space

2.1. **Cartesian coordinate systems.** Every point in a plane can be labeled by an ordered pair of numbers, (x, y) following the rule given in Fig. 3. This was the starting point of Descartes'[1] idea in 1637, and such a coordinate system is named Cartesian after him. Descartes' idea originated what is now called analytic geometry and is one of his everlasting contributions to mathematics. He realized that there is a one-to-one correspondence between curves in a plane and equations in two variables, $f(x, y) = 0$. There is also a such a correspondence between geometric properties of the curve and analytic properties of the function $f(x, y)$. Geometry is thus reduced to algebra and analysis.

In the same way, every point in the space can be labeled by an ordered triple of numbers, (x, y, z), as is shown in Fig. 3. Surfaces in space can be associated to equations in three variables, $F(x, y, z) = 0$. A particular class of such equations have the form $z - f(x, y) = 0$, which represents the graph of a scalar-valued function of two variables, $(x, y) \mapsto f(x, y)$.

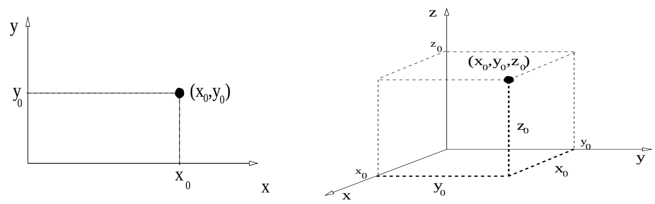


FIGURE 3. Cartesian coordinate systems in two and three dimensions, respectively.

There are two types of coordinates systems in space except for rotations, called **right handed** and **left handed**, and they are pictured in Fig. 4. Except by rotations means that there is no rotation that transforms one coordinate system into the other. More examples of these type of coordinate systems can be seen in

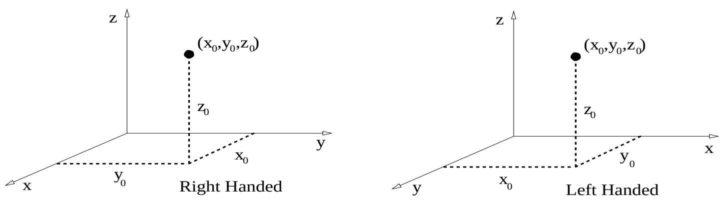


FIGURE 4. Examples of right-handed and left-handed Cartesian coordinate systems in three dimensions, respectively.

Fig. 5, where the first coordinate system is right handed, while the second one is left handed. The coordinate systems on the far left are just the same systems pictured on the far right. The only difference is that they are looked from a different angle. The same type of splitting of the set of all possible Cartesian coordinate systems also happens in $\mathbb{R}^2$, as it can be seen in Fig. 6. The reason to mention this issue

[1] "René Descartes (1596-165, French) is more widely known as a philosopher than as a mathematician, although his philosophy has been controverted while his mathematics has not." Page 138 in [1].

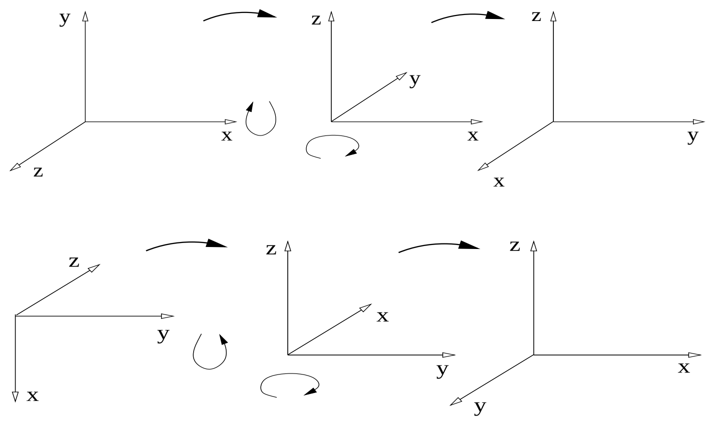


Figure 5. More examples of right-handed and left-handed Cartesian coordinate systems, respectively.

on three dimensional coordinate system the *cross product* of vectors. This is an operation between three dimensional vectors that gives different results in right or left handed coordinate systems. The cross product of two dimensional vectors is not defined. In other to avoid confusion, only right handed coordinates systems will be used in these notes.

Figure 6. Examples of right-handed and left-handed Cartesian coordinate systems, respectively, in two dimensions.

2.2. **Distance formula.** The distance between points in space is crucial to generalize the idea of limit to functions of more than one variable.

Theorem 1. *The distance between the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{2.1}$$

Remark. The proof of the distance formula is based on Pythagoras theorem.

Proof. *(Theorem 1.)* Consider in Fig. 7 the vertical right triangle with vertices at P_1, P_2, Q, and the horizontal right triangle with vertices at P_1, Q, R. Pythagoras theorem in the first triangle says

$$|P_1P_2|^2 = |P_1Q|^2 + |P_2Q|^2,$$

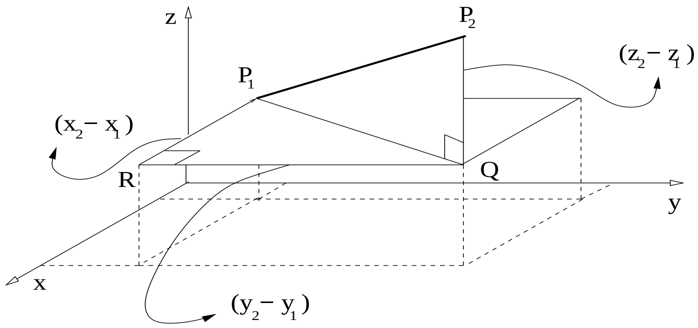


FIGURE 7. Proof of the distance formula between two points in space.

while in the second triangle it says

$$|P_1Q|^2 = |P_1R|^2 + |RQ|^2.$$

Replace the first equation into the second,

$$|P_1P_2|^2 = |P_1R|^2 + |RQ|^2 + |P_2Q|^2,$$

and recall that $|P_1R| = |x_2 - x_1|$, $|RQ| = |y_2 - y_1|$ and $|P_2Q| = |z_2 - z_1|$. Then one obtains

$$|P_1P_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

This formula establishes the Lemma. □

A simple use of the distance formula is to decide whether three points belong to a single line or not. In the first case they are called collinear. If the distance between any pair of points is equal the sum of the distances of the other two possible pair of points, then the points are collinear. Otherwise, they are not.

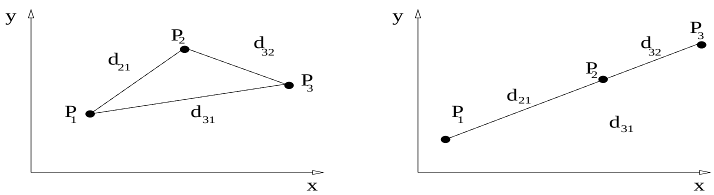


FIGURE 8. In the picture on the left the points are not collinear, and holds $d_{21} + d_{32} > d_{31}$. In the picture on the right the points are collinear, and holds $d_{21} + d_{32} = d_{31}$.

Another simple application of the distance formula is in the definition of a sphere. A set of points $P \in \mathbb{R}^3$ in space is called a **sphere** centered at the point $P_0 \in \mathbb{R}^3$ with radius $R > 0$ iff the points P satisfy the equation $|P_0P| = R$. So, a sphere is the set of points at fixed distance from a center point (see Fig. 9). The distance formula in Eq. (2.1) provides an alternative way to express the equation that satisfy the points $P = (x, y, z)$ in a sphere centered at the point $P_0 = (x_0, y_0, z_0)$ of radius R, which is given by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

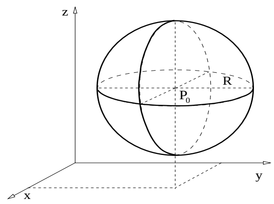


FIGURE 9. Example of a sphere in $\mathbb{R}^3$.

Exercise 1. *Draw the graph of the sphere with equation* $x^2 + (y + 2)^2 + z^2 = 2^2$.

Solution. One must first find the center point P_0 and the radius R of the sphere. From the equation of the sphere it can be seen that the center point is $P_0 = (0, -2, 0)$ and has a radius $R = 2$. Therefore, the graph of the sphere is given in Fig. 10.

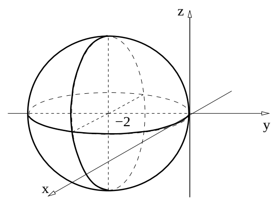


FIGURE 10. The sphere in Exercise 1.

Remark. If one expands the square in $x^2 + (y + 2)^2 + z^2 = 2^2$ one obtains the equivalent expression

$$x^2 + (y + 2)^2 + z^2 = 2^2 \quad \Leftrightarrow \quad x^2 + y^2 + 4y + z^2 = 0.$$

Both equations represent the same sphere, but in the second case is more difficult to see what are the center and the radius. In order to find them out, one has to go backwards, completing the squares.

Exercise 2. *Fix constants a, b, c, and d. Then show that the equation*

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = d$$

is the equation of a sphere iff the condition $d > -(a^2 + b^2 + c^2)$ *holds. Furthermore, give the expressions for the center point* P_0 *and the radius* R *of the sphere in terms of the constants a, b, c, and d.*

2.3. **Vectors on the plane and in space.** The concept of vector is an abstraction that describes many different phenomena. By the beginning of the seventeenth century physicists and mathematicians realized that several different physical phenomena were described using the same idea. These phenomena included velocities, accelerations, forces, rotations, electric and magnetic phenomena, and heat transfer. The idea was of an oriented segment. A **vector** in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is an oriented line segment starting from an initial point and ending at a final point. The initial point is called tail point, and the final point is called head point. See Fig. 11. A vector in a plane or in space is more than a number in the sense that it was needed more than a single number to specify it.

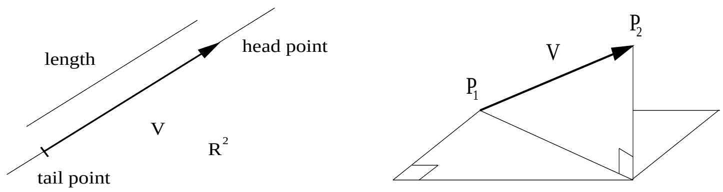


FIGURE 11. Graphical representation of a vector in $\mathbb{R}^2$, and a vector in $\mathbb{R}^3$.

A vector with tail point P_0 and head point P_1 is usually denoted in many different ways. These was include $\overrightarrow{P_0P_1}$, also $\vec{v}$, and $\boldsymbol{v}$. All these notations will be used in these notes. The **length** of a vector is the distance from its tail to head points, and it is denoted by $|\overrightarrow{P_0P_1}|$, or $|\vec{v}|$, and $|\boldsymbol{v}|$.

2.3.1. *Components of a vector.* It can be seen in the definition above that a vector is an object defined independently of any coordinate system. In the case that a Cartesian coordinate system is introduced, a vector can be written in terms of components in this coordinate system. In Fig. 12 it is clear that a vector with tail point $P_1 = (x_1, y_1, z_1)$ and head point $P_2 = (x_2, y_2, z_2)$ has components

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

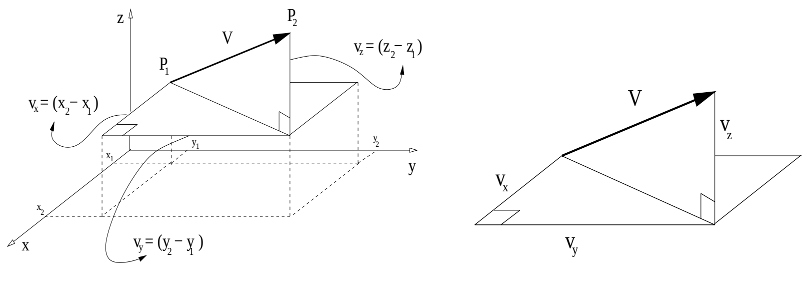


FIGURE 12. Components of a vector in a coordinate system.

Given a vector $\boldsymbol{v}$ in $\mathbb{R}^3$ we will use the notation $\boldsymbol{v} = \langle v_x, v_y, v_z\rangle$ to express this vector in terms of its components in a coordinate system.

The length of a vector $\boldsymbol{v}$, which was defined as the distance from tail to head points, can be expressed in terms of the vector components. The distance formula between two points, Eq. (2.1), implies

$$|\boldsymbol{v}| = \sqrt{(v_x)^2 + (v_y)^2 + (v_z)^2}.$$

Remark. Points and vectors are different objects. This is easy to appreciate when no coordinate system is present. Given a coordinate system, then there is a one-to-one correspondence between points and vectors, where a point P is associated to a vector $\boldsymbol{v} = \overrightarrow{OP}$. That is, a point is associated to a vector with head at that point and its tail at the origin of the coordinate system. Notice that a vector can be translated away from the origin of the coordinate system. Both points and vectors

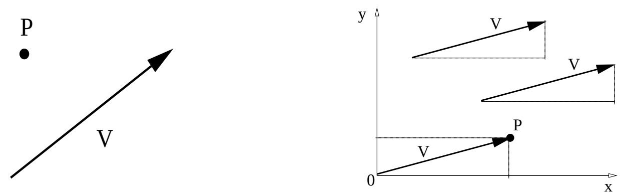


FIGURE 13. Points and vectors are geometrically different objects.

are specified with an ordered pair of numbers in $\mathbb{R}^2$, or an ordered triple of numbers in $\mathbb{R}^3$. To avoid confusion we use round brackets for points and angle brackets for vectors, that is, $(1, 2, 3)$ is a point and $\langle 1, 2, 3\rangle$ is a vector.

Exercise 3. *The vector given in Fig. 14 has length* $|\boldsymbol{v}| = 10$ *and the angle* $\theta = \pi/6$. *Find the components of this vector* $\boldsymbol{v}$ *in the coordinate system represented in that Figure.*

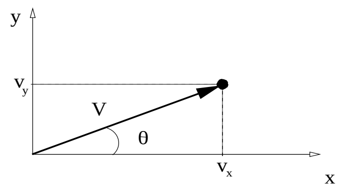


FIGURE 14. Find the components of the vector $\boldsymbol{v}$ in this coordinate system.

Solution. The components of the vector $\boldsymbol{v}$ in the coordinate system defined in Fig. 14 are given by

$$v_x = |\boldsymbol{v}| \cos(\theta) = 10\,\frac{\sqrt{3}}{2}, \qquad v_y = |\boldsymbol{v}| \sin(\theta) = 10\,\frac{1}{2},$$

so the answer is $\boldsymbol{v} = 5\,\langle\sqrt{3}, 1\rangle$.

2.3.2. *Operations with vectors.* The **multiplication** of a vector $\boldsymbol{v}$ by a non-zero number $a \in \mathbb{R}$ is a vector denoted as $a\boldsymbol{v}$, parallel to the vector $\boldsymbol{v}$, pointing in the same direction when $a > 0$ and in opposite direction when $a < 0$, and the length of the vector $a\boldsymbol{v}$ is scaled by $|a|$, that is, $|a\boldsymbol{v}| = |a|\,|\boldsymbol{v}|$. Hence, the resulting vector $a\boldsymbol{v}$ is stretched for $|a| > 1$, is compressed for $|a| < 1$, and their direction reversed for $a < 0$. See Fig. 15.

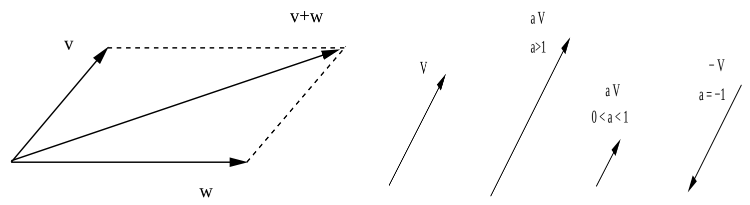


FIGURE 15. The parallelogram law of addition of two vectors, an the multiplication of a vector $\boldsymbol{v}$ by a number a.

The **addition** of two vectors $\boldsymbol{v}$, $\boldsymbol{w}$ is a vector denoted by $\boldsymbol{v} + \boldsymbol{w}$, lying in the plane spanned by the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, with tail and head points given by the parallelogram rule as given in Fig. 15. The **difference** between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is a vector denoted as $\boldsymbol{v} - \boldsymbol{w}$, given by the equation $\boldsymbol{v} - \boldsymbol{w} := \boldsymbol{v} + (-\boldsymbol{w})$. That is, the difference of two vectors is the sum of the first plus the reverse of the second vector. See Fig. 16.

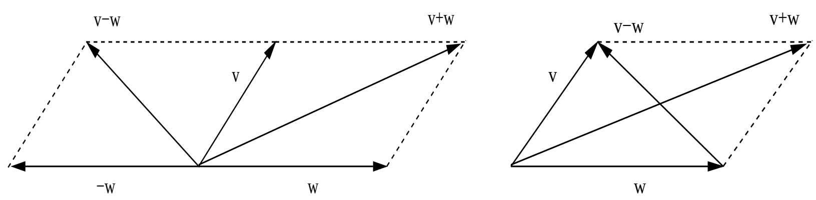


FIGURE 16. The difference between two vectors is a particular case of the addition law.

It is useful to translate these operations with vectors in terms of operations with the vector components. This is summarized in the result below in Theorem 2, in the case of vectors in $\mathbb{R}^2$, and in Theorem 3 in the case of vectors in $\mathbb{R}^3$.

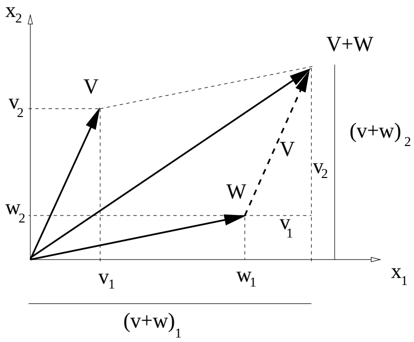
Theorem 2. *Given the vectors* $\boldsymbol{v} = \langle v_x, v_y \rangle$, $\boldsymbol{w} = \langle w_x, w_y \rangle$ *in* $\mathbb{R}^2$, *and a number* $a \in \mathbb{R}$, *then the following expressions hold,*

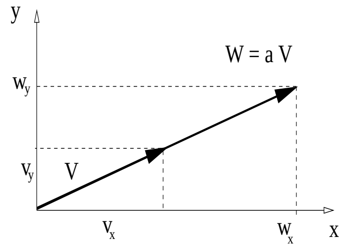
$$\boldsymbol{v} + \boldsymbol{w} = \langle (v_x + w_x), (v_y + w_y) \rangle, \tag{2.2}$$

$$\boldsymbol{v} - \boldsymbol{w} = \langle (v_x - w_x), (v_y - w_y) \rangle, \tag{2.3}$$

$$a\boldsymbol{v} = \langle a v_x, a v_y \rangle. \tag{2.4}$$

Proof. *(Theorem 2.)* In the figure below it is clear that the parallelogram law is equivalent to add the components of the vectors, that is, Eq. (2.2).





The multiplication law, Eq. (2.4), can be obtained from similar triangles in the figure above, that is,

$$\frac{|\boldsymbol{v}|}{v_x} = \frac{|\boldsymbol{w}|}{w_x}, \qquad \frac{|\boldsymbol{v}|}{v_y} = \frac{|\boldsymbol{w}|}{w_y}.$$

Recall that $|\boldsymbol{w}| = a|\boldsymbol{v}|$ and then cancel the factor $|\boldsymbol{v}|$, so one has

$$\frac{1}{v_x} = \frac{a}{w_x}, \qquad \frac{1}{v_y} = \frac{a}{w_y}$$

which implies $w_x = av_x$ and $w_y = av_y$, that is, Eq. (2.4). Finally Eq. (2.3) is straightforward to obtain Eqs. (2.2) and (2.4). □

Analogous expressions hold for vectors in $\mathbb{R}^3$.

Theorem 3. *Given the vectors $\boldsymbol{v} = \langle v_x, v_y, v_z\rangle$, $\boldsymbol{w} = \langle w_x, w_y, w_z\rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the following expressions hold,*

$$\boldsymbol{v} + \boldsymbol{w} = \langle (v_x + w_x), (v_y + w_y), (v_z + w_z)\rangle, \tag{2.5}$$

$$\boldsymbol{v} - \boldsymbol{w} = \langle (v_x - w_x), (v_y - w_y), (v_z - w_z)\rangle, \tag{2.6}$$

$$a\boldsymbol{v} = \langle av_x, av_y, av_z\rangle. \tag{2.7}$$

Proof. *(Theorem 3.)* It is analogous to the proof of Theorem 2 and it is not reproduced here. □

The vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are very useful to write any other vector.

$$\boldsymbol{i} = \langle 1,0,0\rangle, \qquad \boldsymbol{j} = \langle 0,1,0\rangle, \qquad \boldsymbol{k} = \langle 0,0,1\rangle.$$

It can be seen in Fig. 17 that every vector $\boldsymbol{v}$ in $\mathbb{R}^3$ can be written uniquely in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, and the following equation holds, $\mathbf{v} = \langle v_x, v_y, v_z\rangle = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$.

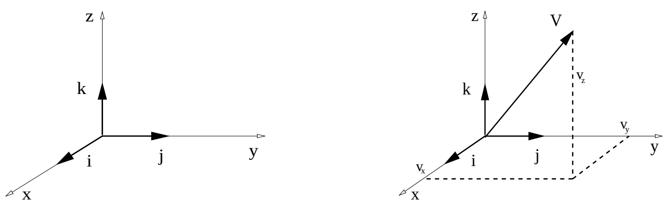
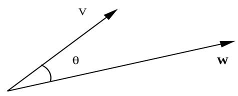


Figure 17. Standard basis vectors in $\mathbb{R}^3$.

2.4. **Dot product and projections.** The dot (also "scalar" or "inner") product of two vectors is a number.

Definition 1. *Let* $\mathbf{v}$, $\mathbf{w}$ *be vectors and* $0 \leqslant \theta \leqslant \pi$ *be the angle in between taken from* $\mathbf{w}$ *to* $\mathbf{v}$. *Then their dot product is given by*

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta),$$



Here are the main properties of the dot product.

Theorem 4. *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors and* a *be a real number. Then,*

(1) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$;
(2) $a(\mathbf{v} \cdot \mathbf{w}) = (a\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (a\mathbf{w})$;
(3) Let $\mathbf{v}$, $\mathbf{w}$ *be nonzero vectors. Then,* $\mathbf{v} \cdot \mathbf{w} = 0 \Longleftrightarrow \mathbf{v} \perp \mathbf{w}$;
(4) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(commutative);*
(5) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(distributive).*

Proof: The first property is straightforward from the definition of dot product, because $\theta = 0$ in this case. The second property is also simple to show, because for $a \geqslant 0$ holds that $a = |a|$, then

$$\begin{aligned} a(\mathbf{v} \cdot \mathbf{w}) = |a|\,|\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) &= |a\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = (a\mathbf{v}) \cdot \mathbf{w}, \\ &= |\mathbf{v}|\,|a\mathbf{w}|\,\cos(\theta) = \mathbf{v} \cdot (a\mathbf{w}). \end{aligned}$$

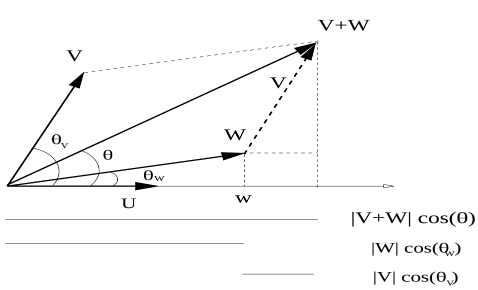
In the case $a < 0$, so $a = -|a|$. Then $a\mathbf{v}$ reverses its direction, then the angle in between $a\mathbf{v}$ and $\mathbf{w}$ is $\pi - \theta$. The same happens with the vectors $\mathbf{v}$ and $a\mathbf{w}$, the angle in between is $\pi - \theta$. Recalling that $\cos(\theta) = -\cos(\pi - \theta)$, one has

$$\begin{aligned} a(\mathbf{v} \cdot \mathbf{w}) = -|a|\,|\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) &= |a\mathbf{v}|\,|\mathbf{w}|\,\cos(\pi - \theta) = (a\mathbf{v}) \cdot \mathbf{w}, \\ &= |\mathbf{v}|\,|a\mathbf{w}|\,\cos(\pi - \theta) = \mathbf{v} \cdot (a\mathbf{w}). \end{aligned}$$

For the third property notice that $\mathbf{v}$ and $\mathbf{w}$ nonzero means that $|\mathbf{v}| \neq 0$ and $|\mathbf{w}| \neq 0$. Then, recalling that $0 \leqslant \theta \leqslant \pi$, one has that

$$0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \quad \Leftrightarrow \quad \cos(\theta) = 0 \quad \Leftrightarrow \quad \theta = \frac{\pi}{2} \quad \Leftrightarrow \quad \mathbf{v} \perp \mathbf{w}.$$

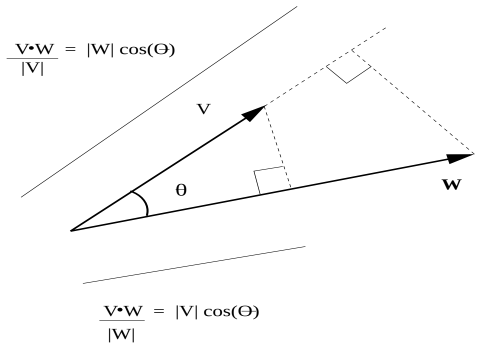
The fourth property comes from the fact that $\cos(\theta)$ is an even function, that is, $\cos(\theta) = \cos(\theta)$. The proof of the fifth property, distributive law for the dot product, is based in the figure below.



$$\left.\begin{aligned} |\mathbf{v}+\mathbf{w}|\cos(\theta) &= \frac{\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})}{|\mathbf{u}|}, \\ |\mathbf{w}|\cos(\theta_w) &= \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}|}, \\ |\mathbf{v}|\cos(\theta_v) &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}, \end{aligned}\right\} \Rightarrow \mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$$

□

The dot product is closely related to projections of one vector onto the other.



Given the vectors $\mathbf{v}$ and $\mathbf{w}$ as in the figure above, the scalar projection of $\mathbf{w}$ onto $\mathbf{v}$, denoted as p_{wv} is given by

$$p_{wv} = |\mathbf{w}|\cos(\theta) = \frac{(\mathbf{v}\cdot\mathbf{w})}{|\mathbf{v}|}.$$

The vector projection of $\mathbf{w}$ onto $\mathbf{v}$, denoted as $\mathbf{p}_{wv}$, is a vector in the direction of $\mathbf{v}$ with length equal to p_{wv}. It is simple to see that

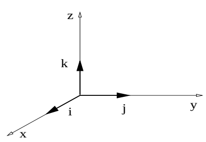
$$\mathbf{p}_{wv} = |\mathbf{w}|\cos(\theta)\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(\mathbf{v}\cdot\mathbf{w})}{|\mathbf{v}|^2}\,\mathbf{v}.$$

The scalar and vector projection of $\mathbf{v}$ onto $\mathbf{w}$ are given by, respectively,

$$p_{vw} = |\mathbf{v}|\cos(\theta) = \frac{(\mathbf{v}\cdot\mathbf{w})}{|\mathbf{w}|},$$

$$\mathbf{p}_{vw} = |\mathbf{v}|\cos(\theta)\frac{\mathbf{w}}{|\mathbf{w}|} = \frac{(\mathbf{v}\cdot\mathbf{w})}{|\mathbf{w}|^2}\,\mathbf{w}.$$

The dot product of the vectors $\mathbf{i} = \langle 1,0,0\rangle$, $\mathbf{j} = \langle 0,1,0\rangle$, and $\mathbf{k} = \langle 0,0,1\rangle$ is very easy to compute, because all have length one, and the angles in between two of them is always $\pi/2$, then



$$\begin{aligned} \mathbf{i}\cdot\mathbf{i} &= 1, & \mathbf{j}\cdot\mathbf{j} &= 1, & \mathbf{k}\cdot\mathbf{k} &= 1, \\ \mathbf{i}\cdot\mathbf{j} &= 0, & \mathbf{j}\cdot\mathbf{i} &= 0, & \mathbf{k}\cdot\mathbf{i} &= 0, \\ \mathbf{i}\cdot\mathbf{k} &= 0, & \mathbf{j}\cdot\mathbf{k} &= 0, & \mathbf{k}\cdot\mathbf{j} &= 0. \end{aligned}$$

2.4.1. *Dot product in components.* The dot product of two vectors can be written in terms of the components of the vectors.

Theorem 5. *Let* $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$. *Then*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

Proof: Recall that $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$, and $\mathbf{w} = w_x \mathbf{i} + w_y \mathbf{j} + w_z \mathbf{k}$, then

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}) \cdot (v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}), \\ &= v_x w_x \, \mathbf{i} \cdot \mathbf{i} + v_y w_y \, \mathbf{j} \cdot \mathbf{j} + v_z w_z \, \mathbf{k} \cdot \mathbf{k}, \\ &= v_x w_x + v_y w_y + v_z w_z, \end{aligned}$$

where the distributive property was introduced to get the second line, together with the dot product of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. □

2.4.2. *Exercises.*

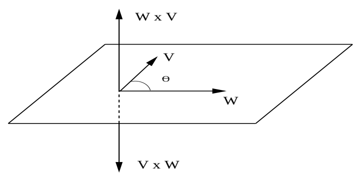
(1) Find the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle -2, 1 \rangle$. ($\theta = \pi/2$.)
(2) Find the scalar and vector projections of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.
The answers are, respectively,

$$|\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = -\frac{2}{\sqrt{5}}; \qquad \begin{aligned} \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} &= -\frac{2}{\sqrt{5}} \frac{1}{\sqrt{5}} \langle 1, 2 \rangle, \\ &= -\frac{2}{5} \langle 1, 2 \rangle . \end{aligned}$$

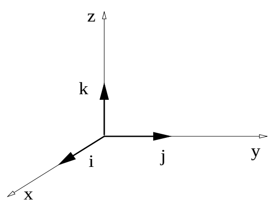
2.5. **Cross product.** The cross product of two vectors is defined for vectors in space. The result is another vector.

Definition 2. *Let* $\mathbf{v}$, $\mathbf{w}$ *be 3-dimensional vectors, and* $0 \leq \theta \leq \pi$ *be the angle in between them from* $\mathbf{w}$ *to* $\mathbf{v}$. *Then,* $\mathbf{v} \times \mathbf{w}$ *is a vector perpendicular to the plane containing* $\mathbf{v}$ *and* $\mathbf{w}$, *pointing in the direction given by the right hand rule*[2], *and with norm*

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}| \, |\mathbf{w}| \sin(\theta).$$



The cross products of the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are simple to compute.



$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{i} &= -\mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}, & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \end{aligned}$$

Notice that the cross product is not associative, that is,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}.$$

This can be seen in the following example,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k}; \quad \text{however} \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = 0.$$

[2] That is, if the right hand fingers swing from $\mathbf{w}$ to $\mathbf{v}$, then the thumb gives the direction of $\mathbf{v} \times \mathbf{w}$.

Therefore the expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ is ambiguous. One has to specify the ordering of the cross products.

Here are the main properties of the cross product.

Theorem 6. *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors and* a *be a real number. Then,*

(1) $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \;\Rightarrow\; \mathbf{v} \times \mathbf{v} = 0$ *(anti-commutative);*
(2) $a(\mathbf{v} \times \mathbf{w}) = (a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w})$*;*
(3) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$*, (distributive law).*

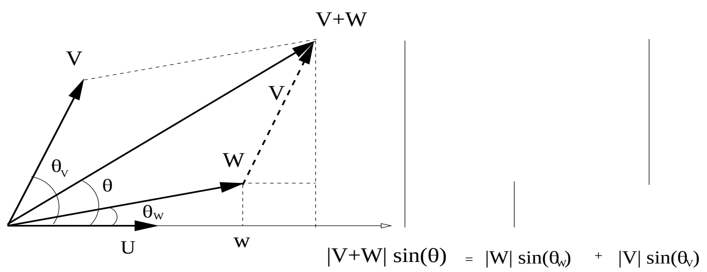
Proof: The first property comes from the right hand rule. For the second property, the case $a = 0$ is trivial, so then consider the case $a > 0$, so $a = |a|$. Then, $a(\mathbf{v} \times \mathbf{w})$ points in the same direction as $\mathbf{v} \times \mathbf{w}$. The length of $a(\mathbf{v} \times \mathbf{w})$ satisfies that,

$$\begin{aligned} |a(\mathbf{v} \times \mathbf{w})| = |a|\,|\mathbf{v}|\,|\mathbf{w}|\,\sin(\theta) = |a\mathbf{v}|\,|\mathbf{w}|\,\sin(\theta) &= |(a\mathbf{v}) \times \mathbf{w}|, \\ &= |\mathbf{v}|\,|a\mathbf{w}|\,\sin(\theta) = |\mathbf{v} \cdot (a\mathbf{w})|. \end{aligned}$$

In the case $a < 0$ the vector $a(\mathbf{v} \times \mathbf{w})$ has the opposite direction of $\mathbf{v} \times \mathbf{w}$. Notice that $a\mathbf{v}$ reverses its direction, then the angle in between $a\mathbf{v}$ and $\mathbf{w}$ is $\pi - \theta$. The right hand rule implies that $(a\mathbf{v}) \times \mathbf{w}$ has also the opposite direction of $\mathbf{v} \times \mathbf{w}$. Precisely the same happens with the vectors $\mathbf{v}$ and $a\mathbf{w}$, the angle in between is $\pi - \theta$, so the he right hand rule implies that $\mathbf{v} \times (a\mathbf{w})$ has also the opposite direction of $\mathbf{v} \times \mathbf{w}$. Recalling that $\sin(\theta) = \sin(\pi - \theta)$, one has

$$\begin{aligned} |a(\mathbf{v} \cdot \mathbf{w})| = |a|\,|\mathbf{v}|\,|\mathbf{w}|\,\sin(\theta) = |a\mathbf{v}|\,|\mathbf{w}|\,\sin(\pi - \theta) &= |(a\mathbf{v}) \cdot \mathbf{w}|, \\ &= |\mathbf{v}|\,|a\mathbf{w}|\,\sin(\pi - \theta) = |\mathbf{v} \cdot (a\mathbf{w})|. \end{aligned}$$

Finally the third property, the distributive property. Consider first the situation given by the figure below.



$$\left.\begin{aligned} |\mathbf{v} + \mathbf{w}|\sin(\theta) &= \frac{|\mathbf{u} \times (\mathbf{v} + \mathbf{w})|}{|\mathbf{u}|}, \\ |\mathbf{w}|\sin(\theta_w) &= \frac{|\mathbf{u} \times \mathbf{w}|}{|\mathbf{u}|}, \\ |\mathbf{v}|\sin(\theta_v) &= \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}|}, \end{aligned}\right\} \Rightarrow |\mathbf{u} \times (\mathbf{v} + \mathbf{w})| = |\mathbf{u} \times \mathbf{v}| + |\mathbf{u} \times \mathbf{w}|$$

In this case, the right hand rule says that the direction of $\mathbf{u} \times \mathbf{v}$ is the same as the direction of $\mathbf{u} \times \mathbf{w}$, which is the same as the direction of $\mathbf{u} \times (\mathbf{v} + \mathbf{w})$. This proves the distributive property for this situation. The other possibilities, for example $\mathbf{u}$ in between $\mathbf{v}$ and $\mathbf{w}$, are left as an exercise. □

A straightforward conclusion from the first property above is the following.

Theorem 7. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be nonzero vectors. Then,* $\mathbf{v} \times \mathbf{w} = 0 \Leftrightarrow \mathbf{v}$ *parallel* $\mathbf{w}$.

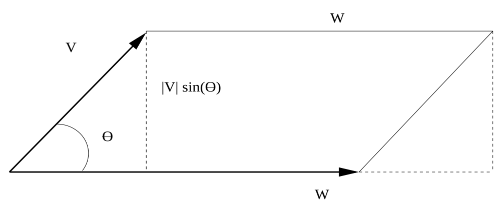
Proof: Both vectors are nonzero, then

$$\begin{aligned} 0 = |\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\,\sin(\theta) \quad &\Leftrightarrow \quad \sin(\theta) = 0, \\ &\Leftrightarrow \quad \theta = 0 \quad \text{or} \quad \theta = \pi, \\ &\Leftrightarrow \quad \mathbf{v} \quad \text{parallel} \quad \mathbf{w}. \end{aligned}$$

The length of the cross product of two vectors vector is the area of the parallelogram formed by these two vectors.

Theorem 8. $|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by* $\mathbf{v}$ *and* $\mathbf{w}$*.*

Proof: This result follows from the figure below.



The area of the parallelogram is the same as the area of the rectangle with sides $|\mathbf{v}|\sin(\theta)$ and $|\mathbf{w}|$ which is precisely $|\mathbf{v}|\,|\mathbf{w}|\sin(\theta) = |\mathbf{v} \times \mathbf{w}|$. $\square$

2.5.1. *The cross product in components.* The cross product of two vectors can be written in terms of the components of the original vectors, as follows.

Theorem 9. *Let* $\mathbf{v} = \langle v_1, v_2, v_3\rangle$*, and* $\mathbf{w} = \langle w_1, w_2, w_3\rangle$*. Then,*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2w_3 - v_3w_2), (v_3w_1 - v_1w_3), (v_1w_2 - v_2w_1)\rangle.$$

Proof: Recall the distributive law for the cross product together with

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Then, one has,

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \times (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}), \\ &= v_1w_2\,\mathbf{i} \times \mathbf{j} + v_1w_3\,\mathbf{i} \times \mathbf{k} + v_2w_1\,\mathbf{j} \times \mathbf{i} + v_2w_3\,\mathbf{j} \times \mathbf{k} + v_3w_1\,\mathbf{k} \times \mathbf{i} + v_3w_2\,\mathbf{k} \times \mathbf{j}, \\ &= (v_2w_3 - v_3w_2)\,\mathbf{j} \times \mathbf{k} + (v_3w_1 - v_1w_3)\,\mathbf{k} \times \mathbf{i} + (v_1w_2 - v_2w_1)\,\mathbf{i} \times \mathbf{j}, \\ &= (v_2w_3 - v_3w_2)\,\mathbf{i} + (v_3w_1 - v_1w_3)\,\mathbf{j} + (v_1w_2 - v_2w_1)\,\mathbf{k}. \end{aligned}$$

$\square$

A possible way to remember the components of a cross product is through the determinant of a 3×3 matrix. Recall first the determinant of a 2×2 matrix,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant of a 3×3 matrix has the form

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

With this expression it is simple to write the components of $\mathbf{v} \times \mathbf{w}$. Writing $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, then

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \\ &= \mathbf{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}, \\ &= (v_2w_3 - v_3w_2)\,\mathbf{i} - (v_1w_3 - v_3w_1)\,\mathbf{j} + (v_1w_2 - v_2w_1)\,\mathbf{k}. \end{aligned}$$

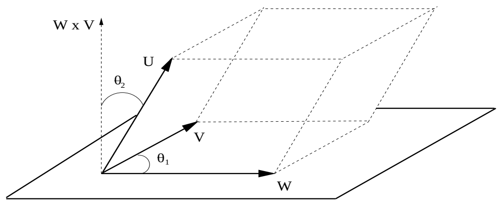
2.5.2. *The triple product of three vectors.*

Definition 3. *The triple product of* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *is the number given by*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

First do the cross product, and then dot the resulting vector with the third vector. The absolute value of the triple product represents the volume of the solid formed with three vectors.

Theorem 10. *Fix nonzero vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. *Then,* $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



Proof: A straightforward computation shows

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}|\,|\cos(\theta_1)|.$$

But this expression is area of the base of the parallelepiped, $|\mathbf{v} \times \mathbf{w}|$, times its height, $|\mathbf{u}||\cos(\theta_1)|$, which is the volume of the parallelepiped. $\square$

Notice that this latter results implies that $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = |(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}|$.

2.5.3. *Exercises.*

(1) Show that given any three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ holds $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

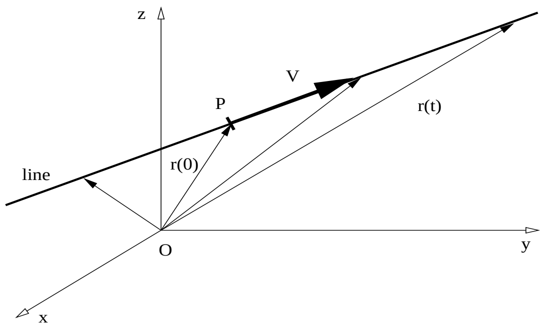
2.6. **Lines and planes in space.**

2.6.1. *Lines in space.* A line is specified by a point in the line and a vector tangent to the line.

Definition 4. *Let* P_0 *be a point in space, and* $\mathbf{v}$ *be a nonzero vector. Fix a coordinate system with origin at* O, *and let* $\mathbf{r}_0 = \overrightarrow{OP}_0$. *Then, the set of vectors*

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}, \tag{2.8}$$

is called the line through P_0 *parallel to* $\mathbf{v}$.



Eq. (2.8) is called the vector equation of the line. We refer to a line to mean both the set of vectors $\mathbf{r}(t)$ and the set of head points of these vectors.

If one writes $\mathbf{r}(t)$ in components, one obtains,

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle,$$
$$\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle,$$
$$\mathbf{v} = \langle v_x, v_y, v_z\rangle,$$

then writing the vector equation as equations for each component one has

$$x(t) = x_0 + tv_x, \tag{2.9}$$
$$y(t) = y_0 + tv_y, \tag{2.10}$$
$$z(t) = z_0 + tv_z. \tag{2.11}$$

Eqs. (2.9)-(2.11) are called the parametric equations of the line.

The parameter t can be computed from the first equation and replaced in the other two equations. Denoting $x = x(t)$, $y = y(t)$, and $z = z(t)$. then,

$$t = \frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}. \tag{2.12}$$

Eq. (2.12) is called the symmetric equations of the line.

Two lines are parallel if their tangent vectors are parallel.

Definition 5. *Two lines $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$, and $\tilde{\mathbf{r}}(s) = \tilde{\mathbf{r}}_0 + s\tilde{\mathbf{v}}$ are parallel if and only if $\mathbf{v} = a\tilde{\mathbf{v}}$, with $a \neq 0$.*

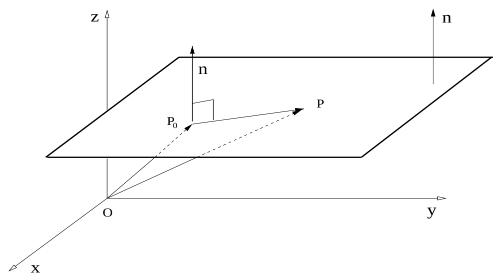
In $\mathbb{R}^2$ two lines are either parallel or they intersect (or both, when they coincide). This is not true in 3 dimensions. Two lines in 3 dimensions are called skew lines if they are neither parallel nor they intersect.

2.6.2. *Planes in space.* A plane in space is determined with a point in the plane and a vector perpendicular to the plane.

Definition 6. *Fix a point in space, P_0, and a nonzero vector $\mathbf{n}$. The set of all points P satisfying*

$$\overrightarrow{P_0P} \cdot \mathbf{n} = 0$$

is called the plane passing through P_0 normal to $\mathbf{n}$, and we denote it as $(P_0, \mathbf{n})$.



So a plane passing through P_0 and normal to $\mathbf{n}$ is the set of all points in space which are the head of a vectors with tail in P_0 and which are perpendicular to $\mathbf{n}$. The condition that two vectors are perpendicular is naturally written in terms of the dot product.

Theorem 11. *Fix a coordinate system with origin at O. The equation of the plane passing through P_0 normal to $\mathbf{n}$ can be written as*

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0,$$

where $\mathbf{r}_0 = \overrightarrow{OP}_0$, and $\mathbf{r} = \overrightarrow{OP}$, with P in the plane.

Proof: Only notice that $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$. Then the theorem follows from the definition of plane. □

The equation of a plane can written explicitly in terms of coordinates.

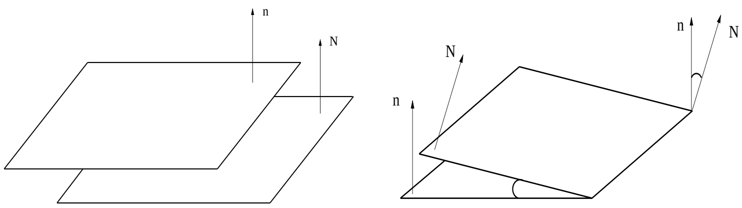
Theorem 12. *In components, the equation of a plane passing through P_0, perpendicular to $\mathbf{n}$ has the form*

$$(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0,$$

where

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \mathbf{n} = \langle n_x, n_y, n_z \rangle, \quad \mathbf{r} = \langle x, y, z \rangle.$$

Definition 7. *Two planes are parallel if their normals are parallel.*

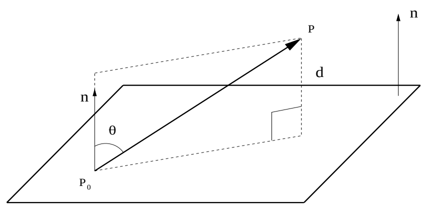


Definition 8. *The angle between two non-parallel planes is the angle between their normal vectors.*

2.6.3. *Distance formula from a point to a plane.*

Theorem 13. *The distance d from a point P to a plane passing through P_0, perpendicular to $\mathbf{n}$ is the shortest distance from P to any point in the plane, and is given by the expression*

$$d = \frac{|\overrightarrow{P_0P} \cdot \mathbf{n}|}{|\mathbf{n}|}. \tag{2.13}$$



Proof: From the figure above one can see that

$$d = |\overrightarrow{P_0P}| \cos(\theta),$$

where θ is the angle between $\overrightarrow{P_0P}$ and $\mathbf{n}$. Recalling that

$$\overrightarrow{P_0P} \cdot \mathbf{n} = |\mathbf{n}|\, |\overrightarrow{P_0P}| \cos(\theta),$$

and that the distance is a nonnegative number, one gets Eq. (2.13). □

2.6.4. *Exercises.*

(1) Are the lines parallel? Do they intersect?

$$x(t) = 1 + t, \qquad x(s) = 2s,$$
$$y(t) = \frac{3}{2} + 3t, \qquad y(s) = 1 + s,$$
$$z(t) = -t, \qquad z(s) = -2 + 4s.$$

(Answer: They are not parallel)
(Answer: They do intersect at $P = \left(1, \frac{3}{2}, 0\right)$.)

(2) Find the equation of the plane that passes through $P = (2, 0, 0)$, $Q = (0, 2, 1)$, $R = (1, 0, 3)$.
(Answer: Find two tangent vectors to the plane, for example, $\overrightarrow{PQ} = \langle -2, 2, 1\rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3\rangle$.
Then, $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 6, 5, 2\rangle$. Pick $P = (2, 0, 0)$.
So the result is $6(x - 2) + 5y + 2z = 0$.)

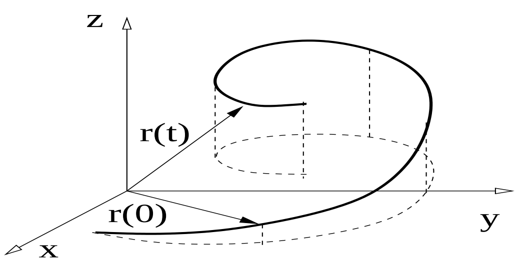
3. Vector-valued functions

3.1. **Definition.** Vector-valued functions are 3 usual functions.

Definition 9. *A vector valued function* $\mathbf{r}(t)$ *on* $\mathbb{R}^3$ *is a function*

$$\mathbf{r} : D \subset \mathbb{R} \to R \subset \mathbb{R}^3,$$

with $n \geq 2$*, the set* D *is called domain of* $\mathbf{r}$*, and* R *is the range of* $\mathbf{r}$*.*



In components, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, where $x(t)$, $y(t)$ and $z(t)$ are usual functions. There is a natural association between curves in $\mathbb{R}^n$ and vector valued functions. The curve is determined by the head points of the vector valued function $\mathbf{r}(t)$. The independent variable t is called the parameter of the curve.

3.1.1. *Limits and continuity.* The limit of $\mathbf{r}(t)$ as $t \to t_0$ is the limit of its components $x(t)$, $y(t)$, $z(t)$.

Definition 10. *Consider the function*

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle.$$

The limit $\lim_{t\to t_0} \mathbf{r}(t)$ *is defined as*

$$\lim_{t\to t_0} \mathbf{r}(t) = \left\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t) \right\rangle,$$

when such limit in each component exists.

Continuous functions are defined in the usual way.

Definition 11. *A function* $\mathbf{r}(t)$ *is continuous at* t_0 *if*

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

A vector-valued function is continuous if the curve representing the function has no holes and no jumps.

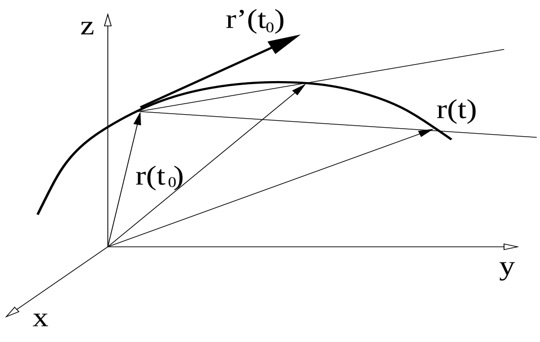
3.1.2. *Derivatives and integrals.* Having the idea of limit, one can introduce the idea of a derivative of a vector valued function. The derivative of a vector valued function $\mathbf{r}(t)$ is another vector valued function, $\mathbf{r}'(t)$.

Definition 12. *The derivative of a vector valued function* $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$*, denoted a* $\mathbf{r}'(t)$*, is given by*

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

when such limit exists.

The derivative, $\mathbf{r}'(t_0)$, of a vector valued function $\mathbf{r}(t)$ is a vector which is tangent to the curve given by $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$.



If $\mathbf{r}(t)$ represents the vector position of a particle, then $\mathbf{r}'(t)$ represents the velocity vector of that particle. That is, $\mathbf{v}(t) = \mathbf{r}'(t)$.

The components of $\mathbf{r}'(t)$ are the derivative of the components of $\mathbf{r}(t)$.

Theorem 14. *Consider the vector-valued function* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$*, where* $x(t)$*,* $y(t)$*, and* $z(t)$ *are differentiable functions. Then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

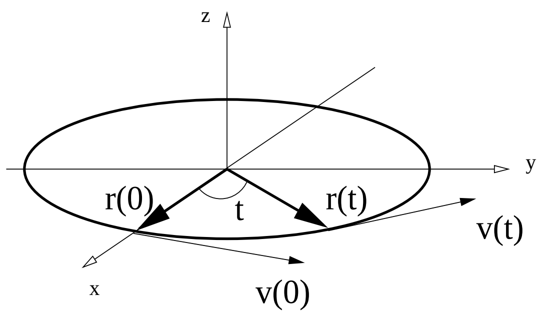
Proof:

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},\\
&= \lim_{h\to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right\rangle\\
&= \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} \right\rangle\\
&= \langle x'(t), y'(t), z'(t)\rangle.
\end{aligned}$$

□

An Example of a vector valued functions and their derivatives is the following:

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle \quad \Leftarrow \quad \mathbf{v}(t) = \langle -\sin(t), \cos(t), 0\rangle.$$



Differentiation rules are the same as for usual (scalar) functions.

Theorem 15. *Let* $\mathbf{v}(t)$ *and* $\mathbf{w}(t)$ *be differentiable vector-valued functions and* $f(t)$ *be a scalar function. Then the following equations hold,*

(1) $[\mathbf{v}(t) + \mathbf{w}(t)]' = \mathbf{v}'(t) + \mathbf{w}'(t)$*, (addition);*
(2) $[c\mathbf{v}(t)]' = c\mathbf{v}'(t)$*, (product rule for constants);*
(3) $[f(t)\mathbf{v}(t)]' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t)$*, (product rule for scalar functions);*
(4) $[\mathbf{v}(f(t))]' = \mathbf{v}'(f(t))f'(t)$*, (chain rule for scalar functions).*

Proof: This theorem follows in a straightforward way if one writes the vector valued functions in components, and uses the usual differentiation rules from scalar functions. □

The derivative of dot and cross product of vectors also satisfy a product rule.

Theorem 16. *Let* $\mathbf{v}(t)$ *and* $\mathbf{w}(t)$ *be differentiable vector-valued functions. Then the following equations hold,*

(1) $[\mathbf{v}(t) \cdot \mathbf{w}(t)]' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t)$*, (product rule for dot product);*
(2) $[\mathbf{v}(t) \times \mathbf{w}(t)]' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t)$*, (product rule for cross product).*

Proof: It is a straightforward but long calculation. Write everything in components.

□

Higher derivatives can also be computed.

Definition 13. *The m derivative of $\mathbf{r}(t)$ is denoted as $\mathbf{r}^{(m)}(t)$ and is given by the expression $\mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]'$.*

Consider the following example:

$$\begin{aligned}
\mathbf{r}(t) &= \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle, \\
\mathbf{r}'(t) &= \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\
\mathbf{r}^{(2)}(t) &= (\mathbf{r}'(t))' = \langle -\cos(t), -\sin(t), 2 \rangle, \\
\mathbf{r}^{(3)}(t) &= (\mathbf{r}^{(2)}(t))' = \langle \sin(t), -\cos(t), 0 \rangle.
\end{aligned}$$

If $\mathbf{r}(t)$ is the vector position of a particle, then the velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t)$, and the acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}^{(2)}(t)$.

Definite integrals are also computed component by component.

Definition 14. *The definite integral of $\mathbf{r}(t)$ form $t \in [a, b]$ is a vector whose components are the integrals of the components of $\mathbf{r}(t)$, namely,*

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle.$$

Consider the following example:

$$\begin{aligned}
\int_0^\pi \langle \cos(t), \sin(t), t \rangle dt &= \left\langle \int_0^\pi \cos(t)dt, \int_0^\pi \sin(t)dt, \int_0^\pi tdt \right\rangle, \\
&= \left\langle \sin(t)\big|_0^\pi, -\cos(t)\big|_0^\pi, \frac{t^2}{2}\bigg|_0^\pi, \right\rangle, \\
&= \left\langle 0, 2, \frac{\pi^2}{2} \right\rangle.
\end{aligned}$$

Finally a brief comment: How to compute derivatives in polar coordinates? The definitions of limit and derivative of vector-valued functions were introduced component by component in a Cartesian coordinate system. What does happen if one needs to work in other coordinate system, like cylindrical or spherical coordinates? All the notions of limit and derivatives for vector-valued functions can be generalized in a way that the Cartesian coordinates system need not to be introduced. When one introduce such Cartesian coordinates systems, one recovers the definitions presented here. We do not study in our course this coordinate independent notion of limit and derivatives.

References

[1] E. Bell. *The development of mathematics.* Dover, New York, 1992. Unabriged an unaltered republication of the second edition, 1945, of the work first published by McGraw-Hill, 1940.