Name: ______________________ **Student Number:** _____________

Math 20C.
Midterm Exam 1
July 9, 2004

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (4 points) Consider the vectors $\vec{v} = 2\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$.

   (a) Compute $\vec{v} \cdot \vec{w}$.

   $$\vec{v} \cdot \vec{w} = \langle 2, -2, 1 \rangle \cdot \langle 1, 2, -1 \rangle = 2 - 4 - 1 = -3.$$

   (b) What is the cosine of the angle between $\vec{v}$ and $\vec{w}$?

   $$|\vec{v}| = \sqrt{4 + 4 + 1} = 3, \quad |\vec{w}| = \sqrt{1 + 4 + 1} = \sqrt{6}.$$
   $$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|\,|\vec{w}|} = \frac{-3}{3\sqrt{6}} = -\frac{1}{\sqrt{6}}.$$

   (c) Find a unit vector in the direction of $\vec{v}$.

   $$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{3}\langle 2, -2, 1 \rangle.$$

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2. (4 points) Find the equation of the plane that contains the lines $\vec{r_1}(t) = \langle 1, 2, 3\rangle t$ and $\vec{r_2}(t) = \langle 1, 1, 0\rangle + \langle 1, 2, 3\rangle t$.

$P_0 = (1, 1, 0)$ is in the plane. $P_1 = (1, 2, 3) = \vec{r_1}(t = 1)$ is also in the plane.

Therefore, $\vec{P_0P_1} = \langle 0, 1, 3\rangle$ is tangent to the plane.

$\vec{v} = \langle 1, 2, 3\rangle$ is also tangent to the plane. Then, the normal vector to the plane $\vec{n}$ can be computed as follows:

$$\vec{n} = \vec{v} \times \vec{P_0P_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{vmatrix} = (6 - 3)\vec{i} - (3 - 0)\vec{j} + (1 - 0)\vec{k} = \langle 3, -3, 1\rangle.$$

Then, the equation of the plane can be constructed with $P_0 = (1, 1, 0)$ and $\vec{n} = \langle 3, -3, 1\rangle$ as follows:

$$3(x - 1) - 3(y - 1) + z = 0,$$

$$3x - 3y + z = 0.$$

3. (4 points) Find an equation for the plane that passes through the points $(2,2,0)$, $(1,0,3)$, and $(0,1,2)$.

Call the points $P=(2,2,0)$, $Q=(1,0,3)$, and $R=(0,1,2)$.

Introduce the vectors $\vec{PR}=\langle -2,-1,2\rangle$, and $\vec{PQ}=\langle -1,-2,3\rangle$.

They are tangent to the plane.

So the normal vector to the plane $\vec{n}$ can be computed as follows:

$$\vec{n}=\vec{PR}\times\vec{PQ}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & 2 \\ -1 & -2 & 3 \end{vmatrix}=(-3-(-4))\vec{i}-(-6-(-2))\vec{j}+(4-1)\vec{k}=\langle 1,4,3\rangle.$$

Then, the equation of the plane passing through $P=(2,2,0)$ with normal $\vec{n}=\langle 1,4,3\rangle$ is

$$(x-2)+4(y-2)+3(z-0)=0,$$

$$x+4y+3z=10.$$

4. (4 points) A particle moves along the curve $\vec{r}(t) = \langle \sin(3t), 4t, \cos(3t) \rangle$, for $t \geq 0$.

(a) Find the velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$ functions of the particle.

$$\begin{aligned} \vec{v}(t) &= \langle 3\cos(3t), 4, -3\sin(3t) \rangle, \\ \vec{a}(t) &= \langle -9\sin(3t), 0, -9\cos(3t) \rangle. \end{aligned}$$

(b) Reparametrize the curve $\vec{r}(t)$ with respect to the arc length measured from the point where $t = 0$, in the direction of increasing t.

$$\begin{aligned} \ell(t) &= \int_0^t |\vec{v}(u)|\, du, \\ &= \int_0^t \sqrt{9\cos^2(3u) + 16 + 9\sin^2(3u)}\, du, \\ &= \int_0^t \sqrt{9+16}\, du, \\ &= 5t. \end{aligned}$$

Then,

$$t = \frac{\ell}{5}.$$

The reparametrized curve $\vec{r}(\ell)$ is then given by

$$\vec{r}(\ell) = \left\langle \sin\left(\frac{3}{5}\ell\right), \frac{4}{5}\ell, \cos\left(\frac{3}{5}\ell\right) \right\rangle.$$