Slide 1

Determinants, $n \times n$

- Review: The 3×3 case.
- Determinants $n \times n$.
  (Expansions by rows and columns.
  Relation with Gauss elimination matrices: Properties.)
- Formula for the inverse matrix.
- Cramer's rule.
  (To solve nonhomogeneous systems of equations.)

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Review: 3×3 determinants can be computed expanding by any row or column

Claim 1 *The determinant of a* 3×3 *matrix can be computed in terms of* 2×2 *determinants, expanding by any column or row, using the following sign convention for the addition,*

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \text{ Sign of coefficient } a_{ij} \text{ is } (-1)^{i+j}.$$

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Review: Main properties of 3×3 determinants

Let $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ be a 3×3 matrix. Let $\mathbf{c}$ be a 3-vector.

- $\det([\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_2]) = 0$.
- $\det([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) = -\det([\mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_3])$.
- $\det([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) = -\det([\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_2])$.
- $\det([c\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_2]) = c\det([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])$.
- $\det([\mathbf{a}_1 + \mathbf{c}, \mathbf{a}_2, \mathbf{a}_3]) = \det([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) + \det([\mathbf{c}, \mathbf{a}_2, \mathbf{a}_3])$.

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Review: Important results concerning 3×3 determinants

Theorem 1 *Let A, B be 3×3 matrices. Then,*

- *A is invertible* $\Leftrightarrow \det(A) \neq 0$.
- $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ *are l.d.* $\Leftrightarrow \det([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) = 0$.
- $\det(A) = \det(A^T)$.
- $\det(AB) = \det(A)\det(B)$.

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Gauss elimination can be used to compute determinants!

Theorem 2 *Let A be a 3×3 matrix.*

- *Let B be the result of adding to a row in A a multiple of another row in A. Then,* $\det(B) = \det(A)$.
- *Let B be the result of interchanging two rows in A. Then,* $\det(B) = -\det(A)$.
- *Let B be the result of multiply a row in A by a number k. Then,* $\det(A) = (1/k)\det(B)$.

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Notation needed for the $n \times n$ case

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & \mathbf{a_{1j}} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \mathbf{a_{i1}} & \cdots & \mathbf{a_{ij}} & \cdots & \mathbf{a_{in}} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & \mathbf{a_{nj}} & \cdots & a_{nn} \end{bmatrix}, \quad \text{eliminate the column } j, \text{ and the row } i.$$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \text{Sign of coefficient } a_{ij} \text{ is } (-1)^{i+j}.$$

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Determinant $n \times n$: expansion by the first row

Definition 1 *The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is given by*

$$\det(A) = \det(A_{11})a_{11} - \det(A_{12})a_{12} + \cdots + (-1)^{1+n}\det(A_{1n})a_{1n},$$

$$= \sum_{j=1}^{n}(-1)^{1+j}\det(A_{1j})\,a_{1j} = \sum_{j=1}^{n} a_{1j}\,C_{1j},$$

where C_{ij} is called the cofactor of a matrix A and is the number given by $C_{ij} = (-1)^{i+j}\det(A_{ij})$.

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Determinants can be computed expanding along any row or any column

Theorem 3 *The determinant of an $n \times n$ matrix $A = [a_{ij}]$ can be computed by an expansion along any row or along any column. That is,*

$$\det(A) = \sum_{j=1}^{n} C_{ij}\,a_{ij}, \quad \text{for any } i = 1, \cdots, n,$$

$$= \sum_{i=1}^{n} C_{ij}\,a_{ij}, \quad \text{for any } j = 1, \cdots, n.$$

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Use the row or column with the most number of zeros to compute the determinant

Theorem 4 *The determinant of a triangular matrix is the product of its diagonal elements.*

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix}(1) = 1 \times 4 \times 6 = 24.$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix}(1) = 1 \times 3 \times 5 = 15.$$

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Main properties of $n \times n$ determinants

Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be a $n \times n$ matrix. Let $\mathbf{c}$ be a n-vector.

- $\det([\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_n]) = 0.$
- $\det([\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n]) = -\det([\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_n]).$
- $\det([\mathbf{a}_1, \cdots, \mathbf{a}_j + \mathbf{c}, \cdots, \mathbf{a}_n]) =$
  $\det([\mathbf{a}_1, \cdots, \mathbf{a}_j \cdots, \mathbf{a}_n]) + \det([\mathbf{a}_1, \cdots, \mathbf{c}, \cdots, \mathbf{a}_n]).$
- $\det([\mathbf{a}_1, \cdots, c\mathbf{a}_j, \cdots, \mathbf{a}_n]) = c\det([\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n]).$

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Important results concerning $n \times n$ determinants

Theorem 5 *Let A, B be $n \times n$ matrices. Then,*

- *A is invertible* $\Leftrightarrow \det(A) \neq 0$.
- $\mathbf{a}_1, \cdots, \mathbf{a}_n$ *are l.d.* $\Leftrightarrow \det([\mathbf{a}_1, \cdots, \mathbf{a}_n]) = 0$.
- $\det(A) = \det(A^T)$.
- $\det(AB) = \det(A)\det(B)$.

The properties of the determinant on the column vectors of A and the property $\det(A) = \det(A^T)$ imply the following results on the rows of A.

Theorem 6 (Determinants and elementary row operations) *Let A be a $n \times n$ matrix.*

- *Let B be the result of adding to a row in A a multiple of another row in A. Then,* $\det(B) = \det(A)$.
- *Let B be the result of interchanging two rows in A. Then,* $\det(B) = -\det(A)$.
- *Let B be the result of multiply a row in A by a number k. Then,* $\det(B) = k\det(A)$.

Determinant and Gauss elimination operations

Theorem 7 *If E represents an elementary row operation and A is an $n \times n$ matrix, then*

$$\det(EA) = \det(E)\det(A).$$

The proof is to compute the determinant of every elementary row operation matrix, E, and then use the previous theorem.

Theorem 8 (Determinant of a product) *If A, B are arbitrary $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

Determinant of a product of matrices

Proof: If A is not invertible, then AB is not invertible, then the theorem holds, because $0 = \det(AB) = \det(A)\det(B) = 0$. Suppose that A is invertible. Then there exist elementary row operations $E_k, \cdots, E_1$ such that

$$A = E_k \cdots E_1.$$

Then,

$$\begin{aligned}
\det(AB) &= \det(E_k \cdots E_1 B), \\
&= \det(E_k)\det(E_{k-1} \cdots E_1 B), \\
&= \det(E_k) \cdots \det(E_1)\det(B), \\
&= \det(E_k \cdots E_1)\det(B), \\
&= \det(A)\det(B).
\end{aligned}$$

□

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Formula for the inverse matrix

Theorem 9 *Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix,* $C_{ij} = (-1)^{i+j}\det(A_{ij})$ *be the* ij*th cofactor, and* $\Delta = \det(A)$*. Then the inverse matrix* A^{-1} *is given by*

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Formula for the inverse matrix *Proof:* It is a straightforward computation. Let us denote B the matrix with components $(B)_{ij} = C_{ji}/\Delta$. Then,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Compute each component of the product AB.

$$(AB)_{11} = \frac{1}{\Delta}(C_{11}a_{11} + C_{12}a_{12} + \cdots + C_{1n}a_{1n}) = 1,$$

because the factor in the numerator in the right hand side is precisely $\det(A) = \Delta$.

The second component is given by

$$(AB)_{12} = \frac{1}{\Delta}(C_{11}a_{21} + C_{12}a_{22} + \cdots + C_{1n}a_{2n}).$$

The factor between brackets in the right hand side is an expansion by the first row of the determinant of a matrix whose first row is

$$a_{21}, a_{22}, \cdots a_{2n}.$$

That is,

$$(AB)_{12} = \frac{1}{\Delta} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

An analogous calculation shows that $(AB)_{ij}$ is given by

$$(AB)_{ij} = \frac{1}{\Delta}(C_{j1}a_{i1} + C_{j2}a_{i2} + \cdots + C_{jn}a_{in}),$$

The factor between brackets in the right hand side is an expansion by the j row of the determinant of a matrix whose j row is is the i row of A,

$$a_{i1}, a_{i2}, \cdots a_{in}.$$

That is,

$$(AB)_{ij} = \frac{1}{\Delta} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{in the } j\text{-row}$$

Therefore, when $i \neq j$ the factor between brackets is the determinant of a matrix with two identical rows, so $(AB)_{ij} = 0$ for $i \neq j$. If $i = j$, the the that factor is precisely $\det(A)$, then $(AB)_{ii} = 1$.

Summarizing,

$$\begin{aligned} (AB)_{ij} &= \frac{1}{\Delta} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{in the } j\text{-row} \\ &= I_{ij} \end{aligned}$$

Repeat this calculation for BA. $\square$

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Cramer's rule is a formula to solve nonhomogeneous linear equations

Theorem 10 *Let A be an invertible $n \times n$ matrix, so the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$.*

Then the components x_i of the solution $\mathbf{x}$ are given by

$$x_i = \frac{1}{\Delta} \det(A_i(\mathbf{b})).$$

where we introduced the matrix $A_i(\mathbf{b}) = [\mathbf{a}_1, \cdots, \mathbf{b}, \cdots, \mathbf{a}_n]$, with $\mathbf{b}$ placed in the i-column.

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Proof: On the one hand, A invertible means that the solution can be written as $\mathbf{x} = A^{-1}\mathbf{b}$. From the formula of the inverse matrix one obtains

$$x_i = \frac{1}{\Delta}(C_{1i}\, b_1 + C_{2i}\, b_2 + \cdots + C_{ni}\, b_n),$$

where b_i are the components of $\mathbf{b}$.

On the other hand, if one expands the $\det(A_i(\mathbf{b}))$ by the i row one gets

$$\det(A_i(\mathbf{b})) = (C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n).$$

$\square$