Slide 1

Linear Transformations

- Domain, range, and null spaces.
- Injective and surjective transformations.
- Bijections and the inverse.
- Nullity + Rank Theorem.
- Components in a basis: Matrices.

Slide 2

Linear transformations are linear functions

Definition 1 *Let V, W be vector spaces. A function $T : V \to W$ is said to be a linear transformation if*

$$T(a\mathbf{u} + b\mathbf{v}) = a\,T(\mathbf{u}) + b\,T(\mathbf{v})$$

for all $\mathbf{u}$, $\mathbf{v} \in V$ and all a, $b \in \mathbb{R}$.

Linear transformations are also called linear functions, linear mappings, or linear operators.

Slide 3

Examples of linear transformations

- The identity transformation, that is $T : V \to V$, given by $T(\mathbf{v}) = \mathbf{v}$.
- A stretching by $a \in I\!R$, that is, $T : V \to V$, given by $T(\mathbf{v}) = a\mathbf{v}$.
- A projection, that is $T : I\!R^3 \to I\!R^3$ given by $T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$.

Slide 4

The derivative is a linear transformation between infinite dimensional vector spaces!

Let V be the space of differentiable functions on $(0, 1)$.

Let W be the space of continuous functions on $(0, 1)$.

Then, $T : V \to W$ given by $T(f) = f'$ is a linear transformation.

Slide 5

The indefinite integral is a linear transformation between infinite dimensional vector spaces

Let V be the space of differentiable functions on $(0, 1)$.

Let W be the space of continuous functions on $(0, 1)$.

Then, $T : W \to V$ given by $T(f) = \int f(x)\,dx$ is a linear transformation.

Slide 6

The domain of T is the set where the function is defined

Definition 2 *The domain of T is denoted as $\mathcal{D}(T)$ and given by*

$$\mathcal{D}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) \text{ is well defined.}\} \subset V.$$

We will study transformations such that $\mathcal{D}(T) = V$.

Slide 7

The range of T is the set generated by T

Definition 3 *The range of T is denoted as $\mathcal{R}(T)$ and given by*

$$\mathcal{R}(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\} \subset W.$$

In general $\mathcal{R}(T)$ is not equal to W.

Slide 8

The null set of T is the set of zeros of T

Definition 4 *The null set of T is denoted as $\mathcal{N}(T)$ and given by*

$$\mathcal{N}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}.\} \subset V.$$

Slide 9

Injective, surjective, and bijective transformations

Definition 5 *Let $T : V \to W$ be a linear transformation. It is said to be*

- *injective (or one-to-one) if for all $\mathbf{v}_1$, $\mathbf{v}_2 \in V$ holds*

$$\mathbf{v}_1 \neq \mathbf{v}_2 \quad \Rightarrow \quad T(\mathbf{v}_1) \neq T(\mathbf{v}_2);$$

- *surjective (or onto) if $\mathcal{R}(T) = W$;*
- *bijective if it is both injective and surjective.*

Slide 10

The null and range sets of a linear transformation are indeed subspaces

Theorem 1 *If $T : V \to W$ is a linear transformation, then $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are subspaces of V and W, respectively.*

Theorem 2 *Let $T : V \to W$ be a linear transformation. T is injective $\Leftrightarrow \mathcal{N}(T) = \{\mathbf{0}\}$.*

Slide 11

Rank and nullity are the dimensions of $\mathcal{N}(T)$ and $\mathcal{R}(T)$

Definition 6 *Let $T : V \to W$ be a linear transformation. The dimensions of $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are said to be the nullity and rank of T, respectively, that is,*

$$\mathrm{rank}(T) = \dim \mathcal{R}(T), \quad \mathrm{null}(T) = \dim \mathcal{N}(T).$$

Slide 12

The fundamental theorem of Linear Algebra relates the dimensions of $\mathcal{N}(T)$, $\mathcal{R}(T)$ and V

Theorem 3 *Let $T : V \to W$ be a linear transformation. If V is finite dimensional, then $\mathcal{R}(T)$ is also finite dimensional and the following relation holds,*

$$\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim V.$$

Proof of Theorem 3: Let $n = \dim V$ and let $S = \{\mathbf{e}_1, \cdots, \mathbf{e}_k\}$ be a basis for $N(T)$, so we say that the nullity is some number $k \geq 0$. Because $N(T)$ is contained in V one knows that $k \leq n$. Let us add l.i. vectors to S to complete a basis of V, say, $\{\mathbf{e}_1, \cdots, \mathbf{e}_k, \mathbf{e}_{k+1}, \cdots, \mathbf{e}_{k+r}\}$ for some number $r \geq 0$ such that $k + r = n$. We shall prove that $\{T(\mathbf{e}_{k+1}), \cdots, T(\mathbf{e}_{k+r})\}$ is a basis for $T(V)$, and then $r = \dim T(V)$. This relation proves Theorem 3.

The elements $\{T(\mathbf{e}_{k+1}), \cdots, T(\mathbf{e}_{k+r})\}$ are a basis of $T(V)$ because they span $T(V)$ and they are l.i.. They span $T(V)$ because for every $\mathbf{w} \in T(V)$ we know that there exists $\mathbf{v} \in V$ such that $\mathbf{w} = T(\mathbf{v})$. If we write $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$, then we have

$$\mathbf{w} = T\left(\sum_{i=0}^{n} \mathbf{e}_i\right) = \sum_{i=0}^{n} a_i T(\mathbf{e}_i) = \sum_{i=k+1}^{k+r} a_i T(\mathbf{e}_i),$$

then the $\{T(\mathbf{e}_{k+1}), \cdots, T(\mathbf{e}_{k+r})\}$ span $T(V)$.

These vectors are also l.i., by the following argument. Suppose there are scalars $c_{k+1}, \cdots, c_{k+r}$ such that

$$\sum_{i=k+1}^{k+r} c_i T(\mathbf{e}_i) = 0.$$

Then, this implies

$$T\left(\sum_{i=k+1}^{k+r} c_i \mathbf{e}_i\right) = 0,$$

so the vector $\mathbf{u} = \sum_{i=k+1}^{k+r} c_i T(\mathbf{e}_i)$ belongs to $N(T)$. But if $\mathbf{u}$ belongs to $N(T)$, then it must be written also as a linear combination of the elements of the base of $N(T)$, namely, the vectors $\mathbf{e}_1, \cdots, \mathbf{e}_k$, so there exists constants $c_1, \cdots, c_k$ such that

$$\mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{e}_i.$$

Then, we can construct the linear combination

$$\mathbf{0} = \mathbf{u} - \mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{e}_i - \sum_{i=k+1}^{k+r} c_i \mathbf{e}_i.$$

Because the set $\{\mathbf{e}_1 \cdots, \mathbf{e}_{k+r}\}$ is a basis o V we have that all the c_i with $1 \leq i \leq k + r$ must vanish. Then, the vectors $\{T(\mathbf{e}_{k+1}), \cdots, T(\mathbf{e}_{k+r})\}$ are l.i.. Therefore they are a basis of $T(V)$, and then the dimension of $\dim T(V) = r$. This proves the Theorem. □

Slide 13

Matrices are the components of a linear transformation in a basis

Definition 7 *Let* $T : V \to W$ *be a linear transformation, where* $\dim V = n$ *and* $\dim W = m$*. Let* $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ *be a basis of* V*. Then,* T *has an associated* $m \times n$ *matrix* A *given by*

$$A = [T(\mathbf{v}_1), \cdots, T(\mathbf{v}_n)].$$

If A is the matrix associated to a linear transformation T, then $T(\mathbf{x}) = A\mathbf{x}$.

It also holds that $\mathcal{N}(T) = N(A)$ and $\mathcal{R}(T) = \mathrm{Col}(A)$.

Slide 14

Here is how to compute the matrix of a linear transformation

Find the matrix associated to $T : \mathbb{R}^2 \to \mathbb{R}^3$, and to the standard bases in $\mathbb{R}^2$ and in $\mathbb{R}^3$, where

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ -x_1 + x_2 \\ x_2 \end{pmatrix},$$

Slide 15

The standard bases are

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \subset I\!R^2,$$

$$\left\{ \mathbf{E}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{E}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{E}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subset I\!R^3.$$

The associated the matrix $A = [T(\mathbf{e}_1), T(\mathbf{e}_2)]$ given by

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Slide 16

The Nullity+Rank theorem can be written in terms of the null and column spaces of a matrix

Theorem 4 *Let A be an $m \times n$ matrix. Then, the following relation holds,*

$$\dim N(A) + \dim \mathrm{Col}(A) = n.$$