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The beginning of Linear Algebra

- Review
  - Broad overview.
  - Main results (so far).
  - Examples.
- The need of abstraction: Vector Space.

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Main results so far including an invertible matrix

Theorem 1 *Let* $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ *be an* $n \times n$ *matrix.*
A is invertible

- $\Leftrightarrow \exists\, A^{-1}$, $n \times n$, *such that* $(A^{-1})A = I = A(A^{-1})$;
- $\Leftrightarrow A\mathbf{x} = \mathbf{b}$ *has a unique solution* $\mathbf{x}$ *for all* $\mathbf{b} \in \mathbb{R}^n$.
- $\Leftrightarrow$ *Col*$(A) = \mathbb{R}^n$;
- $\Leftrightarrow N(A) = \{\mathbf{0}\}$;
- $\Leftrightarrow \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ *are l.i.;*

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Main results so far including an $m \times n$ matrix

Theorem 2 *Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be an $m \times n$ matrix.*

- *$A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in I\!\!R^n \Leftrightarrow \mathbf{b} \in Col(A) \subset I\!\!R^m$.*
- *The solution $\mathbf{x} \in I\!\!R^n$ above is unique $\Leftrightarrow N(A) = \{\mathbf{0}\}$.*

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All these interrelated concepts must be organized in a unified subject

It was realized that the central concept unifying this subject was the one of *linear combination*:

- column vectors: $\mathbf{u} = a\mathbf{v} + \mathbf{w}$;
- $m \times n$ matrices: $C = aA + bB$;
- continuous functions: $h(x) = af(x) + g(x)$;
- row vectors: $\mathbf{u}^T = a\mathbf{v}^T + b\mathbf{w}^T$.

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The operation of linear combination itself was the crucial concept

The elements that are added up or multiplied by a number are NOT important.

This is the breakthrough.

The heart of a Vector Space is the operation of linear combination

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No redundant properties must be specified in the definition of a vector space

Its definition is to specify the operation of linear combination through is properties.

The absence of redundant properties saves much time and effort when one checks whether a set is a vector space.

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A vector space is defined by characterizing the operation linear combination

Definition 1 (Vector space) *Let V be a set of objects, to be called vectors; and let $\mathbb{R}$ be the real numbers. Assume that there are two operations;*

$$(\mathbf{u}, \mathbf{v}) \to \mathbf{u} + \mathbf{v} \in V, \quad (a, \mathbf{v}) \to a\mathbf{v} \in V,$$

called addition and scalar multiplication, respectively, defined for any $\mathbf{u}$, $\mathbf{v} \in V$ and $a \in \mathbb{R}$. These operations are to satisfy the following rules.

(Continues)

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Definition 2 (Vector space, addition)

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *for any* $\mathbf{u}$, $\mathbf{v} \in V$ *(commutative law);*
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, *for any* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$ *(associative law);*
- *there is an element* $\mathbf{0} \in V$ *such that* $\mathbf{0} + \mathbf{u} = \mathbf{u}$ *for any* $\mathbf{u} \in V$ *(existence of the zero element);*
- *for any* $\mathbf{u} \in V$ *there is an element* $-\mathbf{u} \in V$ *such that* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ *(existence of negative elements);*

(Continues)

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Definition 3 (Vector space, multiplication by a number)

- $1\mathbf{u} = \mathbf{u}$ *for any* $\mathbf{u} \in V$*;*
- $a(b\mathbf{u}) = (ab)\mathbf{u}$ *for any* $\mathbf{u} \in V$*,* $a, b \in I\!R$ *(associative law for scalar multiplication);*
- $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ *and* $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ *for any* $\mathbf{u}$*,* $\mathbf{v} \in V$*, and* a*,* $b \in I\!R$ *(distributive laws).*

Then V *is called a real vector space, or a linear space.*

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Known examples of a Vector Space

- Real numbers $I\!R$, with the usual operations $+$ and $\cdot$.
- The space $I\!R^n$ of n-column vectors, or n-tuples ,with the addition and multiplication by real numbers defined in terms of components.
- The set of all $m \times n$ matrices, with the addition of matrices and the multiplication by a number.

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Not so known examples of a Vector Space

- The complex numbers $\mathbb{C}$, with the addition of complex numbers, and multiplication by a real number.
- The set of all polynomials of degree n, $P_n : [a, b] \to I\!R$.
- The set of all polynomials, $P : [a, b] \to I\!R$.
- The space of continuous functions $f : [a, b] \to I\!R$.
- The set of differentiable functions $f : [a, b] \to I\!R$.

From the commutative law and the associative law we observe that to add several elements, the order of the summation does not matter, and it does not cause any ambiguity to write expression such $\mathbf{u} + \mathbf{v} + \mathbf{w}$ or $\Sigma_{i=1}^{n} \mathbf{u}_1$.

By using the commutative law and the associative law it is not difficult to verify that the zero element $\mathbf{0}$ and the negative element $-\mathbf{u}$ of a given element $\mathbf{u} \in V$ are unique, and they can be equivalently defined by the relations $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for any $\mathbf{u} \in V$, and $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$. Below we write $\mathbf{u} - \mathbf{v}$ for $\mathbf{u} + (\mathbf{v})$.

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The zero element is unique

Proof: Suppose there are two elements, $\mathbf{0}_1$, $\mathbf{0}_2$ such that

$$\begin{aligned} \mathbf{0}_1 + \mathbf{u} &= \mathbf{u}, \quad \forall \mathbf{u} \in V, \\ \mathbf{0}_2 + \mathbf{u} &= \mathbf{u}, \quad \forall \mathbf{u} \in V, \end{aligned}$$

then take $\mathbf{u} = \mathbf{0}_2$ in the first equation, and $\mathbf{u} = \mathbf{0}_1$ in the second equation, then one gets, respectively,

$$\mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2, \quad \mathbf{0}_2 + \mathbf{0}_1 = \mathbf{0}_1.$$

The left hand sides are equal, because of the commutative law, the the right hand sides are equal, so $\mathbf{0}_1 = \mathbf{0}_2$. □

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This property is also deduced from the definition of vector space

Theorem 3 $0\mathbf{u} = \mathbf{0}$.

Proof: $\mathbf{u} = 1\mathbf{u} = (0+1)\mathbf{u} = 0\mathbf{u} + 1\mathbf{u} = 0\mathbf{u} + \mathbf{u}$, then

$$0\mathbf{u} + \mathbf{u} = \mathbf{u},$$

that is, $0\mathbf{u}$ is a zero element. But the zero element is unique, so

$$0\mathbf{u} = \mathbf{0}.$$

□

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The negative element is unique

Proof: Assume that there are two negative elements, $\mathbf{v}_1$ and $\mathbf{v}_2$ for $\mathbf{u}$, that is,

$$\mathbf{u} + \mathbf{v}_1 = \mathbf{0}, \quad \mathbf{u} + \mathbf{v}_2 = \mathbf{0}.$$

Then, one has that

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{0} + \mathbf{v}_1, \\ &= \mathbf{u} + \mathbf{v}_2 + \mathbf{v}_1, \\ &= \mathbf{u} + \mathbf{v}_1 + \mathbf{v}_2, \\ &= \mathbf{0} + \mathbf{v}_2, \\ &= \mathbf{v}_2. \end{aligned}$$

$\square$

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Even this property comes from the definition

Theorem 4 $(-\mathbf{u}) = (-1)\mathbf{u}$.

Proof:

$$\begin{aligned} \mathbf{0} &= 0\mathbf{u}, \\ &= (1-1)\mathbf{u}, \\ &= 1\mathbf{u} + (-1)\mathbf{u}, \\ &= \mathbf{u} + (-1)\mathbf{u}, \end{aligned}$$

therefore, $(-1)\mathbf{u}$ is a negative element of $\mathbf{u}$. But negative elements are unique, so,

$$(-1)\mathbf{u} = -\mathbf{u}.$$

$\square$

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Subspaces

- Review: Vector Space.
- Subspace.
- Span.

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A Subspace is a particular class of sets in a vector space

Definition 4 (Subspace) *A subspace W of a vector space V is a subset of V that is closed under the addition and scalar multiplication operations on V.*

That is, $W \subset V$, and for all $\mathbf{u}$, $\mathbf{v} \in W$ and $a \in \mathbb{R}$ holds that

$$\mathbf{u} + \mathbf{v} \in W, \quad a\mathbf{u} \in W.$$

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Examples

- The set $W \subset I\!R^3$ given by

$$W = \{\mathbf{x} \in I\!R^3 : \mathbf{x} = (x_1, x_2, 0), \quad x_1, x_2 \in I\!R\},$$

is a subspace of $I\!R^3$.

- The set $\widehat{W} \subset I\!R^3$ given by

$$\widehat{W} = \{\mathbf{x} \in I\!R^3 : \mathbf{x} = (x_1, x_2, 1), \quad x_1, x_2 \in I\!R\},$$

in contrast is not a subspace of $I\!R^3$.

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Span

Definition 5 *The Span of $\mathbf{v}_1, \cdots \mathbf{v}_n \in V$ is defined as the set of all linear combinations of these vectors, that is*

$$\mathit{Span}(\mathbf{v}_1, \cdots, \mathbf{v}_n) = \{\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n\},$$

where $a_1, \cdots, a_n \in I\!R$.

Theorem 5 *$\mathit{Span}(\mathbf{v}_1, \cdots, \mathbf{v}_n) \subset V$ is a subspace of V.*

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Examples

$$\begin{aligned}\widehat{W} &= \text{span}\{\mathbf{e}_1, \mathbf{e}_2\},\\ &= \text{span}\{(1,1,0),(-1,1,0)\},\\ &= \text{span}\{(2,0,0),(1,1,0),(12,-1,0)\},\\ &= \text{span}\{(2,0,0),(1,1,0),(12,-1,0),(-3,-1,0)\}.\end{aligned}$$

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Examples

- The set $W = \{(x_1, x_2) \in I\!R^2 : x_1 \geq 0\}$ is not a subspace of $I\!R^2$.
- The segment $W\{x \in I\!R : -1 \leq x \leq 1\}$ is not a subspace of $I\!R$.
- The line $W = \{\mathbf{x} \in I\!R^3 : \mathbf{x} = (1,2,3)t\}$ is a subspace or $I\!R^3$.
- The line $W = \{\mathbf{x} \in I\!R^3 : \mathbf{x} = (1,2,0) + (1,2,3)t\}$ is not a subspace or $I\!R^3$.