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A primer on Linear Algebra

- Remarks on the course.
- Overview of Linear Algebra.
- Systems of linear equations.
  (Row approach) (Sec. 1.1).

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Linear Algebra is the study of Vector Spaces

Example of vector spaces: the plane $\mathbb{R}^2$, the space $\mathbb{R}^3$.

Aim of the course:

- To understand the structure of a vector space.
- To use that structure to solve different problems.

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To solve systems of linear equations motivated the creation of Linear Algebra

Plan of the course:

- Solve linear equations.
- Introduce the concept of vectors, matrix, linear transformation.
- Introduce the concept of Vector Space.

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The simplest system of linear equations is a 2×2 system

Example: Find the numbers (x, y) solutions of

$$\begin{aligned} 2x - y &= 0, \\ -x + 2y &= 3. \end{aligned}$$

Row picture: Solve each row separately.

(a) Graphically (lines), (b) By substitution.

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The row picture is appropriate to solve small systems of linear equations

The most general 2×2 system of linear equations is the following: Find (x, y) solution of

$$\begin{aligned} a_{11}x + a_{12}y &= b_1, \\ a_{21}x + a_{22}y &= b_2, \end{aligned}$$

where a_{ij} and b_i are given numbers, with $i = 1, 2$, $j = 1, 2$.

$$\text{Matrix of coeff.} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \quad \text{source} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

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The row picture is appropriate to solve small systems of linear equations

Does a 2×2 system of linear equations have a solution? Is the solution unique?

Graphically one can check that the answer is:

- There exists a unique solution.
  (Lines intersect at a point.)
- There exists infinitely many solutions.
  (Coincident lines.)
- There is no solution.
  (Parallel, non-coincident lines.)

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The row picture is not appropriate to solve big systems of linear equations

Example: Find the numbers (x, y, z) solutions of

$$\begin{aligned} 2x + y + z &= 2, \\ -x + 2y &= 1, \\ x - y + 2z &= -2. \end{aligned}$$

Row picture: Solve each row separately.

(a) Graphically (planes), (b) By substitution.

Check: The solution is $(1, 1, -1)$.

Too complicated.

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Here is the definition of an $m \times n$ system of linear equations

Definition 1 *Fix a set of numbers a_{ij}, b_i, where $i = 1, \cdots, m$ and $j = 1, \cdots, n$. A system of m linear equations in n unknowns x_j, is given by*

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1, \\ \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Consistent: It has solutions (one or infinitely many).
Inconsistent: It has no solutions.

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The matrix of coefficients and the source will be important in the following lectures

The matrix of coefficients, and the source are given by

$$m \text{ rows} \overbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}^{n \text{ columns}} \quad \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$m \times n$ matrix, m column vector.

Next class: Solving Large systems (3×3 or more) in an efficient way: Gauss elimination.

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Gauss elimination

- Review: Systems of linear equations.
- Gauss elimination. (Sec. 1.2)
- Existence and uniqueness of solutions.

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Here is the definition of an $m \times n$ system of linear equations

Definition 2 *Fix a set of numbers a_{ij}, b_i, where $i = 1, \cdots, m$ and $j = 1, \cdots, n$. An $m \times n$ system of m linear equations on n unknowns x_j, is given by*

$$\begin{array}{cccc} a_{11}x_1 + & \cdots & + a_{1n}x_n & = & b_1, \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + & \cdots & + a_{mn}x_n & = & b_m. \end{array}$$

Consistent: It has solutions (one or infinitely many).
Inconsistent: It has no solutions.

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The matrix of coefficients and the source will be important in the following lectures

The matrix of coefficients, and the source are given by

$$m \text{ rows} \overbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}^{n \text{ columns}} = A \qquad \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}$$

$m \times n$ matrix, m column vector.

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The augmented matrix is important is Gauss elimination

The augmented matrix of the former $m \times n$ system is:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & | & b_1 \\ \vdots & & \vdots & | & \vdots \\ a_{m1} & \cdots & a_{mn} & | & b_m \end{bmatrix}$$

Notation: $(A)_{ij} = a_{ij}$, $(\mathbf{b})_i = b_i$ denote the particular coefficients i, j.

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The diagonal elements are also important in Gauss elimination

Definition 3 *The elements $(A)_{ii}$ are called the diagonal elements of A.*

$$\begin{pmatrix} a_{11} & * & * \\ * & a_{22} & * \\ * & * & a_{33} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & * & * \\ * & a_{22} & * \end{pmatrix}, \quad \begin{pmatrix} a_{11} & * \\ * & a_{22} \\ * & * \end{pmatrix}.$$

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Gauss elimination refers to three operations on the augmented matrix

- Add to one row a multiple of the other.
- Interchange two rows.
- Multiply a row by a nonzero constant.

Gauss elimination changes the coefficients of a system but does not change its solution

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The aim of the Gauss elimination is to produce a system whose solutions are easy to read out

- A matrix is in *echelon form* if every element below the diagonal is zero. (Also called upper triangular.)
- A matrix is in *reduced echelon form* if it is in echelon form and the first nonzero element in every row satisfies both
  - it is equal to 1,
  - it is the only nonzero element in that column.

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Gauss elimination makes the following result easy to prove

Theorem 1 (Existence and uniqueness) *A system of linear equations is* inconsistent $\Leftrightarrow$ *the echelon form of the augmented matrix has a row of the form*

$$[0, \cdots, 0|b \neq 0].$$

A consistent *system of linear equations contains either*

- *a unique solution,* (no free variables)*;*
- *or infinitely many solutions,* (at least one free variable)*.*

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The Column Picture: The begin of linear algebra

- Review: The row picture.
- Column picture.
- Vectors, linear combinations, Span.

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Recall the 2×2 system in row picture

Find the numbers (x_1, x_2) solutions of

$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

Row picture: Solve each row separately.

The solution is $x_1 = 1$, and $x_2 = 2$.

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Interpret the 2×2 linear system as an addition of new objects: vectors

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Recall that the solution is $x_1 = 1$ and $x_2 = 2$, that is,

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} 2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

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The solution suggests how to multiply vectors by numbers and how to add vectors

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} 2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix},$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

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The column picture suggests how to multiply vectors by numbers

$$c\mathbf{v} = c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}.$$

The multiplication of a vector by a number stretches or compresses the vector.

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The column picture suggests how to add two vectors

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

The addition of two vectors is represented graphically by the parallelogram law.

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Example 2×2 **revised**

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Find the coefficients x_1, x_2 such that

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 = \mathbf{b},$$

that is, x_1 and x_2 change the length of $\mathbf{a}_1$ and $\mathbf{a}_2$ such that they add up to $\mathbf{b}$.

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The same idea can be generalized to $\mathbb{R}^n$

Vectors **u**, **v** in R^n have the form

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

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The same idea can be generalized to $\mathbb{R}^n$

Addition:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

Multiplication by a number $c \in \mathbb{R}$,

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}.$$

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A linear combination is to add several stretched-compressed vectors

Definition 4 *A vector* $\mathbf{w} \in \mathbb{R}^n$ *is a linear combination of vectors* $\mathbf{v}_1, \cdots, \mathbf{v}_p$ *in* $\mathbb{R}^n$ *if there exist* p *numbers* $c_1, \cdots, c_p \in \mathbb{R}$ *such that*

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Definition 5 *Let* $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$. *The Span*$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ *is the set in* $\mathbb{R}^n$ *formed by of all possible linear combinations of* $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

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Summary of the column picture for linear systems

Does the $m \times n$ linear system below have a solution?

$$\begin{array}{ccccc} a_{11}x_1 + & \cdots & +a_{1n}x_n & = & b_1, \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + & \cdots & +a_{mn}x_n & = & b_m. \end{array}$$

Does the vector $\mathbf{b}$ belong to the Span$\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$?

$$\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \cdots, \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$