Print Name: ______________________ **Student Number:** ______________

Section Time: ______________

Math 20F.
Midterm Exam 2
November 21, 2005

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (6 points) Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

For each of the following expressions, compute it or explain why it is not defined.

(a) $A + A^T$, and $B + B^T$.

(b) AB and BA.

(c) Find a 2×2 matrix C such that $BC = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

(a) $A + A^T$ is not defined, because A is 2×3 and A^T is 3×2.

$$B + B^T = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix}.$$

(b) AB is not defined, because A is 2×3 and B is 2×2.

$$BA = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 9 \\ 0 & 2 & 4 \end{bmatrix}.$$

(c) $\det(B) = 2 - 3 = -1$, then

$$B^{-1} = (-1) \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}.$$

Then,

$$C = B^{-1} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1+6 & -3+12 \\ 1-4 & 3-8 \end{bmatrix},$$

that is,

$$C = \begin{bmatrix} 5 & 9 \\ -3 & -5 \end{bmatrix}.$$

2. (6 points) Find the dimension and a basis for both the null space of A and the column space of A, where

$$A = \begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}.$$

Justify your answers.

$N(A) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$, then we have to solve the homogeneous system $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 5 & 7 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & 6 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $x_4 = 0$, $x_2 + 5x_3 = 0$, $x_1 + 7x_3 = 0$, that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \\ 1 \\ 0 \end{bmatrix} x_3.$$

That is, $\dim N(A) = 1$ and a basis is $\left\{ \begin{bmatrix} -7 \\ -5 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Now, $\dim \mathbb{R}^4 = 4 = \dim N(A) + \dim \operatorname{col}(A) = 1 + \dim \operatorname{col}(A)$, so $\dim \operatorname{col}(A) = 3$. A basis of $\operatorname{col}(A)$ are the three vectors

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \right\}.$$

We only have to show that they are l.i., that is, that the only solution to the equation

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} y_1 + \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} y_2 + \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} y_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is the trivial solution $y_1 = y_2 = y_3 = 0$. This is the case, because,

$$\begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 0 \\ 0 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -3 \\ 0 & -2 & -6 \\ 0 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 3 \\ 0 & 1 & 7 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. (6 points) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be two bases of $I\!R^3$, and suppose that

$$\mathbf{c}_1 = 2\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3, \quad \mathbf{c}_2 = 3\mathbf{b}_2 + \mathbf{b}_3, \quad \mathbf{c}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_3.$$

(a) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Justify your answer.

(b) Consider the vector $\mathbf{x} = \mathbf{c}_1 - 2\mathbf{c}_2 + 2\mathbf{c}_3$. Find $[\mathbf{x}]_{\mathcal{B}}$, that is, the components of $\mathbf{x}$ in the basis $\mathcal{B}$. Justify your answer.

(a)

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = [[\mathbf{c}_1]_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}}] = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \text{and} \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \left(P_{\mathcal{B}\leftarrow\mathcal{C}}\right)^{-1}.$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 0 & -3 & 1 & 0 & 0 \\ -2 & 6 & 0 & 0 & 2 & 0 \\ 2 & 2 & 4 & 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 0 & -3 & 1 & 0 & 0 \\ 0 & 6 & -3 & 1 & 2 & 0 \\ 0 & 2 & 7 & -1 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -3 & 1 & 0 & 0 \\ 0 & 6 & -3 & 1 & 2 & 0 \\ 0 & 6 & 21 & -3 & 0 & 6 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 0 & -3 & 1 & 0 & 0 \\ 0 & 6 & -3 & 1 & 2 & 0 \\ 0 & 0 & 24 & -4 & -2 & 6 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 8 & 0 & -12 & 4 & 0 & 0 \\ 0 & 24 & -12 & 4 & 8 & 0 \\ 0 & 0 & 24 & -4 & -2 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 2 & -1 & 3 \\ 0 & 24 & 0 & 2 & 7 & 3 \\ 0 & 0 & 24 & -4 & -2 & 6 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 24 & 0 & 0 & 6 & -3 & 9 \\ 0 & 24 & 0 & 2 & 7 & 3 \\ 0 & 0 & 24 & -4 & -2 & 6 \end{array}\right],$$

therefore

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \frac{1}{24}\begin{bmatrix} 6 & -3 & 9 \\ 2 & 7 & 3 \\ -4 & -2 & 6 \end{bmatrix}.$$

(b)

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}\,[\mathbf{x}]_{\mathcal{C}} = [[\mathbf{c}_1]_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}}]\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-6 \\ -1-6 \\ 1-2+4 \end{bmatrix},$$

therefore,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}.$$

4. (6 points) For which values of the number a are the following matrices invertible? Justify your answer. Find the inverse whenever is possible.

$$A = \begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & a & 1 \\ 1 & 0 & 1 \\ 1 & -a & -1 \end{bmatrix}.$$

Expand the determinant of A by the third row, then $\det(A) = (-1)(1-a^2)$ so A is invertible for all $a \neq \pm 1$.

Matrix B has two equal columns, the first and the third, therefore, $\det(B) = 0$ for all a, so B is not invertible for all $a \in \mathbb{R}$.

Expand the determinant of C by the second row, then, $\det(C) = -1(-a+a) - 1(a-a) = 0$, therefore C is not invertible for all $a \in \mathbb{R}$.

The inverse of A in the case $a \neq \pm 1$ can be computed in different ways, for example using the method of cofactors, that is,

$$C_{11} = 0, \quad C_{12} = -1, \quad C_{13} = a,$$

$$C_{21} = 0, \quad C_{22} = a, \quad C_{23} = -1,$$

$$C_{31} = (1-a^2), \quad C_{32} = a, \quad C_{33} = -1.$$

Then, the inverse of A is

$$A^{-1} = \frac{1}{(a^2-1)} \begin{bmatrix} 0 & 0 & (1-a^2) \\ -1 & a & a \\ a & -1 & -1 \end{bmatrix}.$$