Print Name: ______________________ **Student Number:** ______________

Math 20F.
Midterm Exam 1
October 17, 2005

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (6 points) Consider the system of linear equations

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & = & 6, \\ -x_1 & - & x_2 & + & 2x_3 & = & -2, \\ x_1 & & & + & 2x_3 & = & 2. \end{array}$$

(a) Use elementary row operations to write the augmented matrix of the system in echelon form.

(b) Find all solutions of the system. If the system has no solutions, explain how you conclude that.

(a)

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 6 \\ -1 & -1 & 2 & -2 \\ 1 & 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ -1 & -1 & 2 & -2 \\ 2 & 3 & -1 & 6 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 0 & -1 & 4 & 0 \\ 0 & 3 & -5 & 2 \end{array}\right] \to$$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 7 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & 2/7 \end{array}\right]$$

(b)

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & 2/7 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2-4/7 \\ 0 & -1 & 0 & -8/7 \\ 0 & 0 & 1 & 2/7 \end{array}\right] = \left[\begin{array}{rrr|r} 1 & 0 & 0 & 10/7 \\ 0 & -1 & 0 & -8/7 \\ 0 & 0 & 1 & 2/7 \end{array}\right]$$

Therefore, the solutions are given by

$$x_1 = \frac{10}{7}, \quad x_2 = \frac{8}{7}, \quad x_3 = \frac{2}{7}.$$

#	Score
1	
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2. (6 points) Find all the solutions of the non-homogeneous system $A\mathbf{x} = \mathbf{b}$, and write them in parametric form, where

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 1 & 8 \\ 1 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 2 & 1 & 8 & 2 \\ 1 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 5 & 10 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

then one has that

$$x_1 = 1 - 3x_3, \quad x_2 = -2x_3, \quad x_3 \text{ is free.}$$

The solution in parametric form is given by

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} x_3.$$

3. (6 points) Consider the matrix A and the vector $\mathbf{b}$ given by

$$A = \begin{bmatrix} 1 & -2 & 7 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

(a) Is $\mathbf{b}$ in the span of the columns of A? Why?

(b) Are the columns of A linearly independent? Why?

(a)

$$\begin{bmatrix} 1 & -2 & 7 & | & 0 \\ 1 & 1 & 1 & | & 1 \\ 2 & 2 & 2 & | & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 7 & | & 0 \\ 0 & 3 & -6 & | & 1 \\ 0 & 6 & -12 & | & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 7 & | & 0 \\ 0 & 3 & -6 & | & 1 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}.$$

The system is inconsistent, so there is no solution. This means that $\mathbf{b}$ is not in the span of the columns of A.

(b) The answer is no, because $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, because the variable x_3 is free, as it can be seen from the last matrix in the solution of part (a).

Just for a double check, we compute these nontrivial solutions as follows:

$$\begin{bmatrix} 1 & -2 & 7 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

Then one has that the nontrivial solutions are

$$x_1 = -3x_3, \quad x_2 = 2x_3, \quad x_3 \text{ is free.}$$

Pick $x_3 = 1$, then one has that

$$-3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix} = 0,$$

so these vectors are linearly dependent.

4. (8 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2 s$ be a linear transformation given by

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Find the matrix A associated to the linear transformation T.

(b) Find $T(-\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$.

(c) Is T one-to-one? Justify your answer.

(d) Is T onto? Justify your answer.

(a)

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)] = \begin{bmatrix} 2 & 3 & -2 \\ 1 & -1 & 2 \end{bmatrix}.$$

(b)

$$T(-\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) = \begin{bmatrix} 2 & 3 & -2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+6-6 \\ -1-2+6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

(c) The answer is no.

T is one-to-one $\Leftrightarrow$ the column vectors of A are linearly independent. However, three vectors in $\mathbb{R}^2$ are always linearly dependent. So, T is not one-to-one.

(c) The answer is yes.

T is onto $\Leftrightarrow$ the column vectors of A span $\mathbb{R}^2$. Two noncollinear vectors in $\mathbb{R}^2$ span the whole space $\mathbb{R}^2$. And, for example, $T(\mathbf{e}_1)$ is not collinear with $T(\mathbf{e}_2)$, so the columns of A span $\mathbb{R}^2$. Then, T is onto.