Slide 1

Matrix operations

- Review:
  - Linear combinations of matrices.
  - Multiplication of matrices.
- Gauss elimination using matrix multiplication.
- Inverse matrix.

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Matrices are a new type of vectors!

Because we can multiply matrices by a number, and we can add matrices (like column vectors),

$$(cA)\mathbf{x} = c(A\mathbf{x}), \quad (A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}.$$

$$\begin{aligned} cA &= [c\mathbf{a}, \cdots, c\mathbf{a}], \\ A + B &= [\mathbf{a}_1 + \mathbf{b}_1, \cdots, \mathbf{a}_n + \mathbf{b}_n]. \end{aligned}$$

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Matrices can also be multiplied (unlike column vectors)

$$\begin{matrix} A & B & \rightarrow & AB \\ m\times n & n\times \ell & & m\times \ell \end{matrix}, \quad (AB)\mathbf{x} = A(B\mathbf{x}).$$

$$AB = [A\mathbf{b}_1, \cdots, A\mathbf{b}_\ell].$$

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Matrix multiplication is very different from number multiplication

- If AB is defined, it does not mean that BA is defined.
- If both AB and BA are defined, it does not mean that AB has the same size that BA.
- If both AB and BA are defined and have the same size, it does not mean that $AB = BA$.

Definition 1 *An $n \times n$ matrix is called a squared matrix.*

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Main properties of the operation with matrices

- $A(BC) = (AB)C$,
- $A(B + C) = AB + AC$,
- $(B + C)A = BA + BC$,
- $a(AB) = (aA)B = A(aB)$,
- $IA = A = AI$.

Notice: For $n \times n$ matrices A, B, one has in general that $AB \neq BA$, that is, the product is not commutative.

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The Gauss elimination operations can be performed by multiplication with matrices

Theorem 1 *Given any $m \times n$ matrix A, denote by B the $m \times n$ matrix in reduced echelon form obtained from A performing k Gauss elimination operations. Then there exist $m \times m$ squared matrices E_1, $\cdots$, E_k, such that*

$$B = E_k \cdots E_1 A.$$

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Some squared matrices have inverse

Definition 2 *A squared matrix A is said to be invertible if there exists a squared matrix, denoted as A^{-1}, satisfying*

$$(A^{-1})A = I, \quad A(A^{-1}) = I.$$

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System of equations wit invertible matrix of coefficients have unique solutions

Theorem 2 *The $n \times n$ matrix A is invertible $\Rightarrow$ The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in I\!R^n$.*

Furthermore, the solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

The converse ($\Leftarrow$) is also true.

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Gauss elimination can be used to compute the inverse matrix

Theorem 3 *Let A be an invertible matrix. Suppose that the Gauss elimination operations $E_1, \cdots, E_k$ transform A into I. Then*

$$A^{-1} = E_k \cdots E_1.$$

Procedure to compute the inverse matrix:

$$[A|I] \to [E_1 A|E_1] \to \cdots \to [E_k \cdots E_1 A|E_k \cdots E_1] = [I|E_k \cdots E_1]$$

then

$$A^{-1} = E_k \cdots E_1.$$