Print Name: ______________________ **Student Number:** ______________

Section Time: ______________

Math 20F.
Midterm Exam 2
March 8, 2006

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (9 points) Consider the linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_2 + 3x_3 \end{bmatrix}, \quad S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ 3x_2 \\ -x_3 \end{bmatrix}$$

(a) Find a matrix A associated to T and the matrix B associated to S. Show your work.

(b) Is T one-to-one? Is T onto? Justify your answer.

(c) Find the matrix of the composition $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$. Justify your answer.

(a) Let $A = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)]$ and $B = [S(\mathbf{e}_1), S(\mathbf{e}_2), S(\mathbf{e}_3)]$. Then,

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -4 \\ 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -4 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 5 \\ 0 & 1 & 3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 3 \\ 0 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, $N(A) = \{\mathbf{0}\}$, and then T is one-to-one.
The relation $\dim N(A) + \dim \mathrm{Col}(A) = 3$ and $\dim N(A) = 0$ imply that $\dim \mathrm{Col}(A) = 3$ and so T is surjective.

(c) The matrix of $T \circ S$ is AB, given by

$$AB = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -3 \\ -2 & 6 & 4 \\ 0 & 3 & -3 \end{bmatrix}.$$

2. (8 points) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be two bases of $I\!R^3$, and suppose that

$$\mathbf{c}_1 = \mathbf{b}_1 - \mathbf{b}_3, \quad \mathbf{c}_2 = 3\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3, \quad \mathbf{c}_3 = \mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3.$$

(a) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$. Justify your answer.

(b) Consider the vector $\mathbf{x} = 2\mathbf{c}_1 + \mathbf{c}_2 - 3\mathbf{c}_3$. Find $[\mathbf{x}]_{\mathcal{B}}$, that is, the components of $\mathbf{x}$ in the basis $\mathcal{B}$. Justify your answer.

(a)

$$[\mathbf{c}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad [\mathbf{c}_2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad [\mathbf{c}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = [[\mathbf{c}_1]_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}}] = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

(b)

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad [\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}},$$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -4 \end{bmatrix}.$$

3. (9 points)

(a) Find the values of the constant s such that $\det(A) = 0$, where,

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & s \\ 1 & s & 3 \end{bmatrix}.$$

(b) Determine the values of s such that the following system of equations below has more than one solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + 3x_2 + sx_3 &= 0 \\ x_1 + sx_2 + 3x_3 &= 0 \end{aligned}$$

(c) Fix $s = 1$ and compute the coefficients $(1,3)$ and $(2,3)$ of A^{-1}. You do not need to compute the rest of the inverse matrix.

(a)

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & s \\ 1 & s & 3 \end{vmatrix}, \\ &= (1)(9 - s^2) - (1)(6 - s) + (-1)(2s - 3), \\ &= 9 - s^2 - 6 + s - 2s + 3, \\ &= -s^2 - s + 6. \end{aligned}$$

Then, $\det(A) = 0$ implies $s^2 + s - 6 = 0$, that is $s = 2$ or $s = -3$.

(b) The same values of s that in (a), because the matrix of coefficients of the linear system is A.

(c) $s = 1$, then $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, and $\det(A) = -1^2 - 1 + 6 = 4$. Also,

$$\left(A^{-1}\right)_{13} = \frac{1}{\det(A)} C_{31}, \quad \left(A^{-1}\right)_{23} = \frac{1}{\det(A)} C_{32}.$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = (3 + 1) = 4,$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -(1 + 2) = -3,$$

then

$$\left(A^{-1}\right)_{13} = 1, \quad \left(A^{-1}\right)_{23} = -\frac{3}{4}.$$

4. (10 points) Consider the matrix $A = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find matrices P and D such that $A = PDP^{-1}$ where P is invertible and D diagonal.

(c) Compute A^3.

(a)

$$0 = \begin{vmatrix} 7-\lambda & 5 \\ 3 & -7-\lambda \end{vmatrix} = -(7+\lambda)(7-\lambda) - 15 = \lambda^2 - 49 - 15 = \lambda^2 - 8^2,$$

then $\lambda_+ = 8$, and $\lambda_- = -8$. The eigenvectors are the following:
For $\lambda_+ = 8$,

$$\begin{bmatrix} -1 & 5 \\ 3 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad x_1 = 5x_2, \quad \Rightarrow \quad \mathbf{x}_+ = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

For $\lambda_- = -8$,

$$\begin{bmatrix} 15 & 5 \\ 3 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad 3x_1 = -x_2, \quad \Rightarrow \quad \mathbf{x}_- = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

(b)

$$P = \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}, \quad P^{-1} = \frac{1}{16}\begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix}.$$

Then, one gets

$$A = \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix} \frac{1}{16} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix}$$

$$A = \frac{1}{2} \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 17 & 10 \\ 6 & -14 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}.$$

(c)

$$A^3 = PD^3P^{-1} = \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 8^3 & 0 \\ 0 & -8^3 \end{bmatrix} \frac{1}{16} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix},$$

$$A = 8^2 \begin{bmatrix} 5 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix} \frac{1}{16} \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix},$$

$$A^3 = 8^2 A.$$