**Print Name:** ______________________ **Student Number:** ______________

**Section Time:** ______________

**Math 20F.**
**Midterm Exam 2**
**March 8, 2006**

*Read each question carefully, and answer each question completely.*
*Show all of your work. No credit will be given for unsupported answers.*
*Write your solutions clearly and legibly. No credit will be given for illegible solutions.*

1. (9 points) Consider the linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_2 + 3x_3 \end{bmatrix}, \quad S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix}$$

   (a) Find a matrix $A$ associated to $T$ and the matrix $B$ associated to $S$. Show your work.

   (b) Is $T$ one-to-one? Is $T$ onto? Justify your answer.

   (c) Find the matrix of the composition $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$. Justify your answer.

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2. (8 points) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be two bases of $\mathbb{R}^3$, and suppose that

$$\mathbf{c}_1 = 3\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3, \quad \mathbf{c}_2 = \mathbf{b}_1 - \mathbf{b}_3, \quad \mathbf{c}_3 = \mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3.$$

(a) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$. Justify your answer.

(b) Consider the vector $\mathbf{x} = 2\mathbf{c}_1 + \mathbf{c}_2 - 3\mathbf{c}_3$. Find $[\mathbf{x}]_{\mathcal{B}}$, that is, the components of $\mathbf{x}$ in the basis $\mathcal{B}$. Justify your answer.

3. (9 points)

(a) Find the values of the constant $k$ such that $\det(A) = 0$, where,

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix}.$$

(b) Determine the values of $k$ such that the following system of equations below has more than one solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + 3x_2 + kx_3 &= 0 \\ x_1 + kx_2 + 3x_3 &= 0 \end{aligned}$$

(c) Fix $k = 1$ and compute the coefficients $(1, 2)$ and $(2, 3)$ of $A^{-1}$. You do not need to compute the rest of the inverse matrix.

4. (10 points) Consider the matrix $A = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}$

(a) Find the eigenvalues and eigenvectors of $A$.

(b) Find matrices $P$ and $D$ such that $A = PDP^{-1}$ where $P$ is invertible and $D$ diagonal.

(c) Compute $A^3$.