Print Name: ______________________ **Student Number:** _____________

Section Time: _____________

Math 20F.
Final Exam
March 20, 2006

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 1 & 3 & -3 \end{bmatrix}$.

   (a) (5 Pts.) Find a basis for the subspace of all vectors $\mathbf{b}$ such that the linear system $A\mathbf{x} = \mathbf{b}$ has solutions. Show your work.

   (b) (5 Pts.) Find a basis for the null space of A. Show your work.

   (c) (5 Pts.) Find a solution to the linear system $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$.

   Is this solution unique? If yes, say why. If no, find a second solution $\mathbf{x}$ with the same $\mathbf{b}$.

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2. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $T : I\!R^2 \to I\!R^2$ be a linear transformation given by

$$T(\mathbf{u}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad T(\mathbf{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(a) (5 Pts.) Find the matrix $A = [T(\mathbf{e}_1), T(\mathbf{e}_2)]$ of the linear transformation, where $\mathbf{e}_1 = \frac{1}{2}(\mathbf{u} + \mathbf{v})$ and $\mathbf{e}_2 = \frac{1}{2}(\mathbf{u} - \mathbf{v})$. Show your work.

(b) (5 Pts.) Compute the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. Compute also the area of the parallelogram formed by $T(\mathbf{u})$ and $T(\mathbf{v})$. Show your work.

3. Let $T : I\!R^3 \to I\!R^2$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + 3x_3 \\ -3x_1 + x_2 \end{bmatrix}.$$

(a) (3 Pts.) Find the matrix A associated to the linear transformation T using the standard bases in $I\!R^3$ and $I\!R^2$. Show your work.

(b) (5 Pts.) Find a basis for the column space of A. Show your work.

(c) (5 Pts.) Is T one-to-one? Is T onto? Justify your answer.

4. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$ given by

$$\mathbf{u}_1 = 2\mathbf{e}_1 - 9\mathbf{e}_2, \quad \mathbf{u}_2 = \mathbf{e}_1 + 8\mathbf{e}_2,$$

where $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbb{R}^2$.

(a) (5 Pts.) Find both change of basis matrices $P_{\mathcal{U}\leftarrow\mathcal{E}}$ and $P_{\mathcal{E}\leftarrow\mathcal{U}}$. Show your work.

(b) (5 Pts.) Consider the vector $\mathbf{x} = 2\mathbf{u}_1 + \mathbf{u}_2$. Find $[\mathbf{x}]_{\mathcal{E}}$, that is, the components of $\mathbf{x}$ in the standard basis. Show your work.

5. (5 Pts.) Consider the matrix

$$A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}.$$

Find the coefficients $(A^{-1})_{13}$ and $(A^{-1})_{21}$ of the inverse matrix of A. You do not need to compute the rest of the inverse matrix. Show your work.

6. Let k be any number, and consider the matrix A given by

$$A = \begin{bmatrix} 2 & -2 & 4 & -1 \\ 0 & 3 & k & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (3 Pts.) Find the eigenvalues of A, and their corresponding multiplicity. Show your work.

(b) (7 Pts) Find the number k such that there exists an eigenspace $E_A(\lambda)$ that is two dimensional, and find a basis for this $E_A(\lambda)$. The notation $E_A(\lambda)$ means the eigenspace corresponding to the eigenvalue λ of matrix A. Show your work.

7. (a) (5 Pts.) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Show your work.

   (b) (3 Pts.) Find matrices P and D such that $A = PDP^{-1}$, where P is invertible an D diagonal. Show your work.

8. Consider the subspace $W \subset I\!R^3$ given by

$$W = \text{Span}\left\{ \mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

(a) (5 Pts.) Find an ortho**normal** basis of W using the Gram-Schmidt process starting with the vector $\mathbf{u}_1$. Show your work.

(b) (5 Pts.) Decompose the vector $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ as follows, $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{x}'$, with $\hat{\mathbf{x}} \in W$ and $\mathbf{x}'$ perpendicular to any vector in W. Show your work.