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Functions defined using infinite series

- Power series.
- The convergence of power series.
- Differentiation and integration.

Slide 2

The geometric series can be used to define a function

We have learned how to add infinitely many terms.

We can use this knowledge to define functions.

$$f(x) = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

In this case we know the explicit expression for the sum:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

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A power series is an infinite sum of power functions

Definition 1 *A power series centered at* $x = 0$ *is given by*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

A power series centered at $x = a$ *is given by*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where a, c_n *are constants.*

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Not every function constructed with an infinite series is a power series

Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which converges for $p > 1$.

(By integral test, although the number that it converges to is not know exactly.)

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^x, \quad x \in (1, \infty),$$

converges, but it is not a power series.

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Here is a simple example of a power series

$$f(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n, \quad 0 < x < 4.$$

Show that for $0 < x < 4$ holds

$$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots,$$

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What is the interval in x where $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges?

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\, x^2 + 3!\, x^3 + \cdots.$$

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Summary about the convergence of power series

Theorem 1 *The convergence of* $\sum_{n-0}^{\infty} c_n (x-a)^n$ *is described by one of the following cases:*

- *Exists* $R > 0$ *such that the series converges for* $|x-a| < R$ *and diverges for* $|x-a| > R$. *The series may or may not converge at the endpoints* $x = a + R$ *and* $x = a - R$.
- *The series converges for all* $x \in \mathbb{R}$ $(R = \infty)$.
- *The series converges only for* $x = a$ $(R = 0)$.

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We introduce here the radius and interval of convergence

In the above formulas R is called the radius of convergence.

The interval of radius R centered at $x = a$ where the series converges is called interval of convergence.

This interval can be open, closed, of half open.

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Differentiation and integration of power series is done term by term

Theorem 2 *Suppose that* $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *converges for* $|x-a| < R$*, with* $R > 0$*. Then,*

$$f'(x) = \sum_{n=0}^{\infty} n\, c_n (x-a)^{n-1},$$

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{c_n}{(n+1)} (x-a)^{(n+1)} + c.$$

Both $f'(x)$ *and* $\int f(x)\, dx$ *converges for* $|x-a| < R$*.*