

## The geometric series can be used to define a function

We have learned how to add infinitely many terms. We can use this knowledge to define functions.

Slide 2

$$f(x) = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

In this case we know the explicit expression for the sum:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

A power series is an infinite sum of power functions

**Definition 1** A power series centered at x = 0 is given by

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

Slide 3

A power series centered at 
$$x = a$$
 is given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

where a,  $c_n$  are constants.

## Not every function constructed with an infinite series is a power series

Consider the *p*-series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ , which converges for p > 1.

Slide 4

(By integral test, although the number that it converges to is not know exactly.)

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^x, \quad x \in (1,\infty),$$

converges, but it is not a power series.

Here is a simple example of a power series

$$f(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n, \quad 0 < x < 4.$$

Slide 5

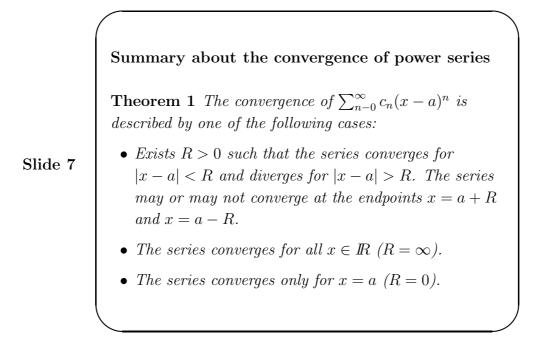
Show that for 0 < x < 4 holds

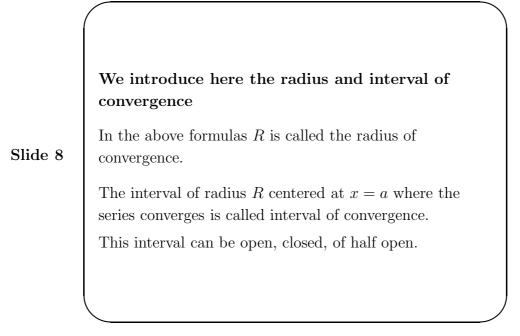
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$$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots,$$

What is the interval in x where  $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges?

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,$$
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$
$$\sum_{n=0}^{\infty} n! \, x^n = 1 + x + 2! \, x^2 + 3! \, x^3 + \cdots.$$





Differentiation and integration of power series is done term by term

**Theorem 2** Suppose that  $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for |x-a| < R, with R > 0. Then,

Slide 9

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1},$$
$$\int f(x) \, dx = \sum_{n=0}^{\infty} \frac{c_n}{(n+1)} (x-a)^{(n+1)} + c.$$

Both f'(x) and  $\int f(x) dx$  converges for |x - a| < R.