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Criteria for convergence of series

- Review: A rough criteria.
- Integral test for convergence.
- Estimating the remainder.
- Comparison test for convergence.

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Here is a very rough criteria to check if a series diverges

Theorem 1 *If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.*

It is based in the following result:

Theorem 2 *If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.*

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Here is a test of convergence for certain series using integration

Theorem 3 *Let $f(x)$ be a continuous, positive, decreasing function on $[1, \infty)$. Consider the sequence $a_n = f(n)$, $n \in \mathbb{Z}^+$. Then,*

$\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow \int_1^{\infty} f(x) dx$ is convergent.

The proof is to show the following inequality

$$\int_1^{n+1} f(x) dx \leq s_n \leq a_1 + \int_1^n f(x) dx.$$

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We already used this test in the harmonic series

The harmonic series $s_n = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges, because

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x} dx \leq s_n,$$

and $\ln(n+1)$ diverges when $n \rightarrow \infty$.

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This test can be applied to the geometric series

Consider the geometric series

$$s_n = \sum_{k=1}^n r^{(k-1)} = 1 + r + r^2 + r^3 + \dots + r^{n-1},$$

for $0 < r < 1$.

Here $f(x) = r^x$. Recall $\int r^x dx = \frac{r^x}{\ln(r)} + c$. Then,

$$s_n \leq 1 + \int_1^n r^x dx = 1 + \frac{r}{\ln(1/r)} - \frac{r^n}{\ln(1/r)}.$$

and $r^n \rightarrow 0$ when $n \rightarrow \infty$. So, s_n converges.

We knew that the geometric series converges, $\sum_{k=1}^{\infty} r^{(k-1)} = \frac{1}{1-r}$.

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This test can be applied to the p -series

Theorem 4 *The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, and it diverges for $p \leq 1$.*

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

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The remainder of a series can be estimated with the same type of integral bounds as above

Definition 1 If $s_n = \sum_{k=1}^n a_k$ converges to s , then the remainder after n terms is

$$R_n = s - s_n.$$

So, $R_n = a_{n+1} + a_{n+2} + \cdots$, that is,

$$R_n = \sum_{k=n+1}^{\infty} a_k.$$

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Here is the estimation for the remainder

Theorem 5 Let $f(x)$ be continuous, positive, decreasing function on $[n, \infty)$. Consider the sequence $a_k = f(k)$ for $k \geq n$. Suppose that $\sum_{k=1}^{\infty} a_k$ is convergent. Then,

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx.$$

Therefore,

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx.$$

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There are three types of comparison tests**Theorem 6** *Let $\sum_{n=1}^{\infty} a_n$, and $\sum_{n=1}^{\infty} b_n$ be series with positive terms.*

1. *If $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *If $\lim_{n \rightarrow \infty} a_n/b_n = c > 0$, finite, then both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ converge, or both diverge.*