Slide 1

Integrals of functions on infinite domains

- Review: Improper integrals type I.
- Type II: Three main possibilities.
- Limit of an infinite sequence.

Slide 2

Generalizations of $\int_a^b f(x)\,dx$ in $I = [a, b]$

Integrals on infinite domains are called improper integrals of type I

- Type I: The interval is infinite: $I = (-\infty, b]$, or $I = [a, \infty)$ or $I = (-\infty, \infty)$.

Integrals of divergent functions on finite domains are called improper integrals of type II.

- Type II: $f(x)$ is not bounded at one or more points in $[a, b]$. ($f(x)$ can have a vertical asymptote in $[a, b]$.)

Slide 3

Type II: Vertical asymptote at b

Possibility (a):

Definition 1 *If $f(x)$ is continuous in $[a, b)$ then*

$$\int_a^{b^-} f(t)\,dt = \lim_{x\to b^-} \int_a^x f(t)\,dt.$$

The integral is said to converge if the limit exists and it is finite.

Otherwise the integral is said to diverge.

Slide 4

Type II: Vertical asymptote at a

Possibility (b):

Definition 2 *If $f(x)$ is continuous in $(a, b]$ then*

$$\int_{a^+}^{b} f(t)\,dt = \lim_{x\to a^+} \int_x^b f(t)\,dt.$$

The integral is said to converge if the limit exists and it is finite.

Otherwise the integral is said to diverge.

Slide 5

Type II: Vertical asymptote in the interior

Possibility (c):

Definition 3 *If $f(x)$ has a vertical asymptote at $c \in (a, b)$, then*

$$\int_a^b f(t)\,dt = \int_a^{c^-} f(t)\,dt + \int_{c^+}^b f(t)\,dt$$

provided that both integrals in the right hand side are convergent.

Slide 6

Comparison theorem: Type I (a) case

Theorem 1 *Let $f(x)$, $g(x)$ be continuous functions for $x \geq a$ and such that $0 \leq g(x) \leq f(x)$. Then:*

- *If $\int_a^\infty f(x)\,dx$ converges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ converges.*
- *If $\int_a^\infty g(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ diverges.*

There are analogous versions for all other cases.

Slide 7

A sequence is a function whose domain are the positive integers

Definition 4 *If for every positive integer* n *there is associated a real or complex number* a_n*, then the ordered set*

$$a_1, a_2, \cdots, a_n, \cdots$$

is called an infinite sequence. Is is denoted as $\{a_n\}$*.*

Definition 5 *a function* $f : \mathbb{Z}^+ \to \mathbb{R}$ *(or* $\mathbb{C}$*) is called an infinite sequence.*

Slide 8

The limits of sequences is the same as in functions of real numbers

Definition 6 *The sequence* $\{a_n\}$ *is said to have the limit* L *is for all* $\epsilon > 0$ *there exists a number* $N > 0$ *such that*

$$|a_n - L| < \epsilon, \quad \textit{for all } n \geq N.$$

In this case we say $\lim_{n\to\infty} a_n = n$ or $a_n \to L$ as $n \to \infty$. We say that the sequence converges.

Otherwise, we say that the sequence diverges.

Slide 9

Increasing-decreasing and bounded above-below are important classes of sequences

- A sequence $\{a_n\}$ is said to be increasing $\Leftrightarrow a_n < a_{n+1}$ for all $n \geq 1$.

  A sequence $\{a_n\}$ is said to be decreasing $\Leftrightarrow a_{n+1} < a_n$ for all $n \geq 1$.

- A sequence $\{a_n\}$ is said to be bounded above $\Leftrightarrow$ exists $M > 0$ such that $a_n < M$ for all $n \geq 1$.

  A sequence $\{a_n\}$ is said to be bounded below $\Leftrightarrow$ exists $m > 0$ such that $m < a_n$ for all $n \geq 1$.

Slide 10

Important tool to show that a sequence converges

- If $\{a_n\}$ is increasing and bounded above then converges.
- If $\{a_n\}$ is decreasing and bounded below then converges.