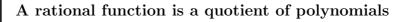


- Review: Decomposition of a polynomial.
- Slide 1
- Integration of rational functions: Cases I IV.
- Examples.
- The three main integrals.



$$\frac{P_n(x)}{Q_m(x)} = S_p(x) + \frac{R_q(x)}{Q_m(x)}$$

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with p = n - m and  $0 \le q < m$ .

Quotients of polynomials can always be integrated

They can be reduced into one of four cases

## There are four possible cases in the decomposition of a rational function

I: The denominator is a product of distinct linear factors.

II: The denominator is a product of linear factors, some of which are repeated.

III: The denominator contains irreducible quadratic factors, none of which are repeated.

IV: The denominator contains irreducible quadratic factors, some of which are repeated.

Case I: The denominator is a product of distinct linear factors

$$\int \frac{2x^2 + 5x - 1}{x(x-1)(x+2)} \, dx.$$

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Case II: The denominator is a product of linear factors, some of which are repeated

$$\int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2} \, dx$$

Case III: The denominator contains irreducible quadratic factors, none of which are repeated

$$\int \frac{3x^2 + 2x - 2}{(x - 1)(x^2 + x + 1)} \, dx$$

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Case IV: The denominator contains irreducible quadratic factors, some of which are repeated

$$\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x - 1)(x^2 + 2)^2} \, dx$$

Every polynomial can be decomposed into a product of polynomials of degree one and two Theorem 1 Every polynomial  $Q_m(x)$  with  $m \ge 0$  and real coefficients can be decomposed as  $Q_m(x) = a(x-a_1)^{\ell_1} \cdots (x-a_r)^{\ell_r} (x^2 + b_1 x + c_1)^{m_1} \cdots (x^2 + b_s x + c_s)^{m_s}$ . with  $m = \ell_1 + \cdots + \ell_r + m_1 + \cdots + m_s$ . The  $a_1, \cdots, a_r$  are roots of  $Q_m(x)$ , that is,  $Q_m(a_i) = 0$ , for  $i = 1, \cdots, r$ .

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The problem of integrate a rational function reduces to that of calculating integrals of the form:

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$$I_{1} = \int \frac{dx}{(x+a)^{m}},$$
$$I_{2} = \int \frac{x \, dx}{(x^{2}+bx+c)^{m}}, \quad I_{3} = \int \frac{dx}{(x^{2}+bx+c)^{m}},$$

The solution for  $I_1$  is:

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$$I_1 = \int \frac{dx}{(x+a)^m} = \begin{cases} \ln(|x+a|) + c & m = -1, \\ \frac{1}{(1-m)(x+a)^{m-1}} + c & m \neq -1/ \end{cases}$$

The integrals  $I_2$  and  $I_3$  can be transformed into the following:

$$\tilde{I}_2 = \int \frac{u \, du}{(u^2 + \alpha^2)^m}, \quad \tilde{I}_3 = \int \frac{du}{(u^2 + \alpha^2)^m}.$$

with the substitution

$$u = x + \frac{b}{2}, \quad \alpha = \frac{1}{2}\sqrt{4c - b^2}.$$

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Slide 10  $\tilde{I}_{2} = \int \frac{u \, du}{(u^{2} + \alpha^{2})^{m}} = \begin{cases} \frac{1}{2} \ln(u^{2} + \alpha^{2}) + c & m = -1, \\ \frac{1}{2(1-m)(u^{2} + \alpha^{2})^{m-1}} + c & m \neq -1/ \end{cases}$  Slide 11

The solution of  $\tilde{I}_3$  is:  $\tilde{I}_3 = \int \frac{du}{(u^2 + \alpha^2)} = \frac{1}{\alpha} \arctan\left(\frac{u}{\alpha}\right) + c, \quad m = 1.$ The case m > 1 is reduced to the case m = 1 by the following recursive formula  $\int \frac{du}{(u^2 + \alpha^2)^m} = \frac{1}{2(m-1)\alpha^2} \frac{u}{(u^2 + \alpha^2)^m} + \frac{2m-3}{2(m-1)\alpha^2} \int \frac{du}{(u^2 + \alpha^2)^{(m-1)}}.$ So, we have integrated all possible cases!