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Integration of rational functions

- Review: Decomposition of a polynomial.
- Integration of rational functions: Cases I - IV.
- Examples.
- The three main integrals.

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A rational function is a quotient of polynomials

$$\frac{P_n(x)}{Q_m(x)} = S_p(x) + \frac{R_q(x)}{Q_m(x)}.$$

with $p = n - m$ and $0 \leq q < m$.

Quotients of polynomials can always be integrated

They can be reduced into one of four cases

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There are four possible cases in the decomposition of a rational function

I: The denominator is a product of distinct linear factors.

II: The denominator is a product of linear factors, some of which are repeated.

III: The denominator contains irreducible quadratic factors, none of which are repeated.

IV: The denominator contains irreducible quadratic factors, some of which are repeated.

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Case I: The denominator is a product of distinct linear factors

$$\int \frac{2x^2 + 5x - 1}{x(x-1)(x+2)}\, dx.$$

Case II: The denominator is a product of linear factors, some of which are repeated

$$\int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2}\, dx$$

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Case III: The denominator contains irreducible quadratic factors, none of which are repeated

$$\int \frac{3x^2 + 2x - 2}{(x-1)(x^2+x+1)}\,dx$$

Case IV: The denominator contains irreducible quadratic factors, some of which are repeated

$$\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2}\,dx$$

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Every polynomial can be decomposed into a product of polynomials of degree one and two

Theorem 1 *Every polynomial $Q_m(x)$ with $m \geq 0$ and real coefficients can be decomposed as*

$$Q_m(x) = a(x-a_1)^{\ell_1}\cdots(x-a_r)^{\ell_r}\left(x^2 + b_1\,x + c_1\right)^{m_1}\cdots\left(x^2 + b_s\,x + c_s\right)^{m_s}.$$

with $m = \ell_1 + \cdots + \ell_r + m_1 + \cdots + m_s$.

The $a_1, \cdots, a_r$ are roots of $Q_m(x)$, that is, $Q_m(a_i) = 0$, for $i = 1, \cdots, r$.

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The problem of integrate a rational function reduces to that of calculating integrals of the form:

$$I_1 = \int \frac{dx}{(x+a)^m},$$

$$I_2 = \int \frac{x\,dx}{(x^2+bx+c)^m}, \quad I_3 = \int \frac{dx}{(x^2+bx+c)^m}.$$

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The solution for I_1 is:

$$I_1 = \int \frac{dx}{(x+a)^m} = \begin{cases} \ln(|x+a|) + c & m = -1, \\ \frac{1}{(1-m)(x+a)^{m-1}} + c & m \neq -1/ \end{cases}$$

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The integrals I_2 and I_3 can be transformed into the following:

$$\tilde{I}_2 = \int \frac{u\,du}{(u^2+\alpha^2)^m}, \quad \tilde{I}_3 = \int \frac{du}{(u^2+\alpha^2)^m}.$$

with the substitution

$$u = x + \frac{b}{2}, \quad \alpha = \frac{1}{2}\sqrt{4c - b^2}.$$

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The solution of $\tilde{I}_2$ is:

$$\tilde{I}_2 = \int \frac{u\,du}{(u^2+\alpha^2)^m} = \begin{cases} \frac{1}{2}\ln(u^2+\alpha^2) + c & m = -1, \\ \frac{1}{2(1-m)(u^2+\alpha^2)^{m-1}} + c & m \neq -1/ \end{cases}$$

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The solution of $\tilde{I}_3$ is:

$$\tilde{I}_3 = \int \frac{du}{(u^2+\alpha^2)} = \frac{1}{\alpha}\arctan\left(\frac{u}{\alpha}\right) + c, \quad m = 1.$$

The case $m > 1$ is reduced to the case $m = 1$ by the following recursive formula

$$\int \frac{du}{(u^2+\alpha^2)^m} = \frac{1}{2(m-1)\alpha^2}\frac{u}{(u^2+\alpha^2)^m} + \frac{2m-3}{2(m-1)\alpha^2}\int \frac{du}{(u^2+\alpha^2)^{(m-1)}}.$$

So, we have integrated all possible cases!