Slide 1

$z = a + bi$, powers, roots, and exponentials

- Review: Cartesian and polar representations.
- Powers and roots.
- Exponential and Euler formula.

Slide 2

Complex numbers can be associated with points in a plane

- Cartesian picture: Good for representing addition and real number multiplication.
  (Parallelogram law and stretching.)
- Polar picture: Good for representing the multiplication law.
  (Stretching and rotation.)

Slide 3

The power of a complex number is very easy to compute in the polar representation

Theorem 1 (De Moivre)

$$(r[\cos(\theta) + i\sin(\theta)])^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Equivalently:

$$z = r[\cos(\theta) + i\sin(\theta)] \Rightarrow z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Slide 4

Arbitrary powers are easy in polar representation

$$z = a + bi, \quad \Leftrightarrow \quad z = r[\cos(\theta) + i\sin(\theta)],$$

$$r = \sqrt{a^2 + b^2}, \quad \theta = \arctan(b/a).$$

Then,

$$(a + bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Slide 5

Magic at work: There are n solutions to the n-th root of a complex number

(In real numbers there are one or two, for n is odd or even, respectively.)

Theorem 2 *Let $z = r[\cos(\theta) + i\sin(\theta)]$ and $n \geq 1$. Then, the complex numbers*

$$w_k = r^{\frac{1}{n}} \left[\cos\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right) \right]$$

$k = 0, \cdots, n-1$ *satisfy the equation*

$$(w_k)^n = z.$$

Slide 6

First, recall a simple result with real numbers

Theorem 3

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n,$$

for all $x \in I\!R$.

Slide 7

Second, consider the following application of De Moivre formula

$$[\cos(\theta) + i\sin(\theta)] = \left[\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right]^n.$$

Therefore,

$$[\cos(\theta) + i\sin(\theta)] = \lim_{n\to\infty}\left(1 + \frac{i\theta}{n}\right)^n.$$

Slide 8

Euler formula is one of the most beautiful formulas we have seen so far

The calculation above *suggests* the following relation:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

In particular, one has Euler formula:

$$e^{i\pi} - 1 = 0.$$