Name: ______________________ **Section Number:** ______

TA Name: ______________________ **Section Time:** ______

Math 20B.
Midterm Exam 2
March 1, 2006

Turn off and put away your cell phone.
No calculators or any other devices are allowed on this exam.
You may use one page of notes, but no books or other assistance on this exam.
Read each question carefully, answer each question completely, and show all of your work.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

1. (6 points) Evaluate $\displaystyle\int e^{i2x}\cos(x)\,dx$. You may leave the result in exponential form.

$$\begin{aligned}\int e^{i2x}\cos(x)\,dx &= \frac{1}{2}\int e^{i2x}\left(e^{ix}+e^{-ix}\right)\,dx,\\ &= \frac{1}{2}\int\left(e^{i3x}+e^{ix}\right)\,dx,\\ &= \frac{1}{2}\left(\frac{1}{3i}e^{i3x}+\frac{1}{i}e^{ix}\right),\\ &= -\frac{i}{6}\left(e^{i3x}+3e^{ix}\right).\end{aligned}$$

2. (6 points) Evaluate $\displaystyle\int \frac{1}{(x^2+16)^2}\,dx.$

Introduce the substitution $x = 4\tan(\theta)$, then $dx = 4\sec^2(\theta)d\theta$, and $x^2+4^2 = 4\sec^2(\theta)$. Therefore,

$$
\begin{aligned}
\int \frac{dx}{(x^2+4^2)^2} &= \int \frac{1}{(4^2\sec^2(\theta))^2}\,4\sec^2(\theta)\,d\theta,\\
&= \frac{1}{4^3}\int \frac{d\theta}{\sec^2(\theta)},\\
&= \frac{1}{4^3}\int \cos^2(\theta)\,d\theta,\\
&= \frac{1}{(2)4^3}\int [1+\cos(2\theta)]\,d\theta,\\
&= \frac{1}{2^7}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + c,\\
&= \frac{1}{2^7}\left(\theta + \sin(\theta)\cos(\theta)\right) + c.
\end{aligned}
$$

Recall

$$\cos(\theta) = \frac{1}{1+\tan^2(\theta)} = \frac{1}{\sqrt{1+(x/4)^2}} = \frac{4}{\sqrt{x^2+4^2}},$$

and

$$\sin(\theta) = \sqrt{1-\cos^2(\theta)} = \frac{x}{\sqrt{x^2+4^2}}.$$

Then, the integral is

$$\int \frac{dx}{(x^2+4^2)^2} = \frac{1}{2^7}\left[\frac{4x}{x^2+4^2} + \arctan\left(\frac{x}{4}\right)\right] + c.$$

3. (8 points) Evaluate $\displaystyle\int \frac{2x^4+3x^2-4}{x^3+x}\,dx.$

$$\frac{2x^4+3x^2-4}{x^3+x} = 2x + \frac{x^2-4}{x^3+x}$$

$$\frac{x^2-4}{(x^2+1)x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}, \quad \Rightarrow \quad x62-4 = A(x_1^2) + (Bx+C)x,$$

then $A=-4$, $B=5$, and $C=0$.

$$\frac{2x^4+3x^2-4}{x^3+x} = 2x - \frac{4}{x} + \frac{5x}{x^2+1},$$

then the integral is:

$$\begin{aligned}\int \frac{2x^4+3x^2-4}{x^3+x}\,dx &= \int 2x\,dx - \int \frac{4}{x}\,dx + \int \frac{5x}{x^2+1}\,dx,\\ &= x^2 - 4\ln(|x|) + \frac{5}{2}\int \frac{du}{u},\\ &= x^2 - 4\ln(|x|) + \frac{5}{2}\ln(x^2+1) + c.\end{aligned}$$

4. (6 points) A radar gun was used to record the speed (in meters per second) of a runner during the first six seconds of a race; the data is recorded in the table below. Estimate the distance the runner traveled during the six seconds using the Trapezoid Rule. (Note: You need not simplify the resulting expression.)

Time (sec)	0	2	4	6
Velocity (meters/sec)	0	8.3	9.3	10.3

Let $V(t)$ be the velocity as function of time.

$$
\begin{aligned}
T_3 &= \left[\frac{1}{2}V(0) + V(2) + V(4) + \frac{1}{2}V(6)\right], \\
&= [2(8.3) + 2(9.3) + 10.3].
\end{aligned}
$$

5. (a) (2 points) $\int_0^\infty e^{-x}\,dx$ converges. Find its value.

$$\int_0^\infty e^{-x}\,dx = \lim_{x\to\infty}\int_0^x e^{-t}\,dt = \lim_{x\to\infty} -(e^{-x}-1) = 1.$$

Therefore,

$$\int_0^\infty e^{-x}\,dx = 1.$$

(b) (4 points) By applying the Comparison Theorem, determine whether

$$\int_0^\infty e^{\cos(x)}\cdot e^{-x}\,dx$$

converges. Be sure to justify your conclusion.

$$-1 \le \cos(x) \le 1 \quad\Rightarrow\quad \frac{1}{e} \le e^{\cos(x)} \le e, \quad\Rightarrow\quad 0 \le e^{\cos(x)}\,e^{-x} \le e\,e^{-x},$$

Therefore,

$$0 \le \int_0^\infty e^{\cos(x)}e^{-x}dx \le e\int_0^\infty e^{-x}dx = e,$$

which says that the integral $\int_0^\infty e^{\cos(x)}e^{-x}dx$ converges.

6. (6 points) Determine whether the sequence $\left\{\frac{2n^2}{5n^2+2n+1}\right\}$ converges. If it converges, find its limit.

$$a_n = \frac{2n^2}{5n^2+2n+1} = \frac{2}{5+\frac{2}{n}+\frac{1}{n^2}} \to \frac{2}{5}, \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} \frac{2n^2}{5n^2+2n+1} = \frac{2}{5}.$$