Name: ______________________________ **Section Number:** ______

TA Name: ______________________________ **Section Time:** ______

Math 20B.
Midterm Exam 1
February 1, 2006

Turn off and put away your cell phone.
No calculators or any other devices are allowed on this exam.
You may use one page of notes, but no books or other assistance on this exam.
Read each question carefully, answer each question completely, and show all of your work.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

1. (6 points) Evaluate the following integrals.

(a) $\displaystyle\int 3x\sin(x^2)\,dx$

Substitute $u = x^2$, then $du = 2x\,dx$, so,

$$\int 3x\sin(x^2)\,dx = \frac{3}{2}\int \sin(u)\,du = -\frac{3}{2}\cos(u) + c, \quad \Rightarrow$$

$$\Rightarrow \quad \int 3x\sin(x^2)\,dx = -\frac{3}{2}\cos(x^2) + c.$$

(b) $\displaystyle\int_2^3 x^2\sqrt{x-2}\,dx$

Substitute $u = x - 2$, then $du = dx$, so,

$$\int_2^3 x^2\sqrt{x-2}\,dx = \int_0^1 (u+2)^2\sqrt{u}\,du = \int_0^1 (u^2+4u+4)\sqrt{u}\,du,$$

$$\int_2^3 x^2\sqrt{x-2}\,dx = \int_0^1 \left(u^{5/2} + 4u^{3/2} + 4u^{1/2}\right)du = \left[\frac{2}{7}u^{7/2} + 4\,\frac{2}{5}u^{5/2} + 4\,\frac{2}{3}u^{3/2}\right]\Bigg|_0^1,$$

so the answer is

$$\int_2^3 x^2\sqrt{x-2}\,dx = \frac{2}{7} + \frac{8}{5} + \frac{8}{3}.$$

2. (8 points) Let $\mathcal{R}$ be the region enclosed by the curves $y = x^2$ and $y = 4$.

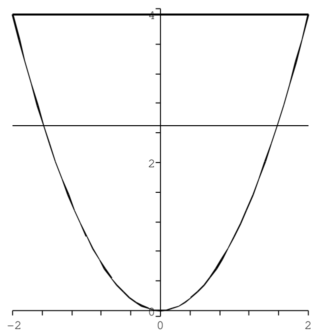
(a) Find the area of the region $\mathcal{R}$.

$$A = \int_{-2}^{2} (4 - x^2)\, dx = \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2} = 16 - \frac{2}{3}8$$

so the answer is

$$A = \frac{32}{3}.$$

(b) Find the number b such that the line $y = b$ divides the region $\mathcal{R}$ in part (a) into two regions with equal area. [Hint: Try integrating with respect to y rather than x.]



The function $y = x^2$ can be written as $x = \sqrt{y}$. So the total area computed above is

$$A = 2\int_0^4 \sqrt{y}\, dy.$$

Then, the number b such that the area up to b is $A/2$ is determined by the equation

$$\frac{16}{3} = 2\int_0^b \sqrt{y}\, dy,$$

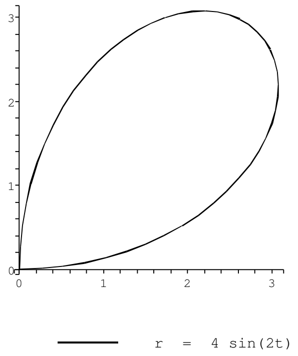
and the solution is:

$$\frac{16}{3} = 2\int_0^b y^{1/2}\, dy = 2\,\frac{2}{3}\left(y^{3/2}\right)\Big|_0^b = \frac{4}{3}b^{3/2},$$

then

$$b^{3/2} = 4 \quad \Rightarrow \quad b = 2^{4/3}.$$

3. (6 points) Find the area enclosed by one loop of the polar curve $r = 4\sin(2\theta)$.



The variable θ belongs to the interval $[0, \pi/2]$. Then the area enclosed by the curve is

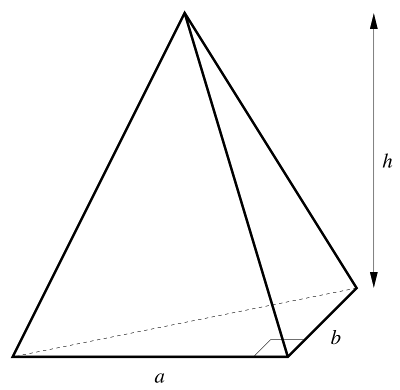
$$A = \int_0^{\pi/2} \frac{1}{2}[r(\theta)]^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} 16\sin^2(2\theta) = 8\int_0^{\pi/2} \frac{1}{2}[1 - \cos(2\theta)] \, d\theta$$

$$A = 4\left(\frac{\pi}{2} - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/2}\right) = 2\pi.$$

So the answer is

$$A = 2\pi.$$

4. (6 points) Find the volume of a tetrahedron with height h and with a right triangular base with side lengths a and b. [Note: A tetrahedron is a pyramid with a triangular base.]



Introduce a coordinate system with the z axis parallel to the height of the tetrahedron and $z = 0$ at its base. Let the x-axis be parallel to the side with length a, and the y-axis be parallel to the side with length b.

The volume of the tetrahedron is

$$V = \int_0^h A(z)\, dz,$$

where z is the coordinate in the direction of the height, and $A(z)$ is the area of the cross sections perpendicular to z. The area is: $A = (1/2)xy$ where x and y are the coordinates of the cross-sectiontriangle at $z \in [0, h]$.

By similar triangles, one can see that

$$\frac{h}{z} = \frac{a}{x}, \qquad \frac{h}{z} = \frac{b}{y}, \qquad \Rightarrow \qquad x = \frac{a}{h} z, \qquad y = \frac{b}{h} z.$$

Therefore,

$$A(z) = \frac{1}{2} \frac{ab}{h^2} z^2.$$

The volume is

$$V = \frac{1}{2} \frac{ab}{h^2} \int_0^h z^2\, dz = \frac{1}{2} \frac{ab}{h^2} \frac{1}{3} h^3 = \frac{1}{6}\, abh,$$

so the answer is $V = abh/6$.

5. (8 points) Let $z = 1 + \sqrt{3}i$.

(a) Write z in polar form.

$$|z| = \sqrt{1+3} = 2, \quad z = |z|e^{1\theta} = |z|(\cos(\theta) + i\sin(\theta),$$

then

$$1 = 2\cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

$$\sqrt{3} = 2\sin(\theta) \quad \Rightarrow \quad \sin(\theta) = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Then,

$$z = 2e^{i\pi/3}.$$

(b) Find z^{10} and write it in standard $(a + bi)$ form.

$$z^{10} = 2^{10}e^{i10\pi/3}.$$