Slide 1

Orthogonal vectors, spaces and bases

- Review: Inner product → Norm → Distance.
- Orthogonal vectors and subspaces.
- Orthogonal projections.
- Orthogonal and orthonormal bases.

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An inner product fixes the notions of angles, length and distance

(,), must be positive, symmetric and linear, that is,

1. $(\mathbf{u}, \mathbf{u}) \geq 0$, and $(\mathbf{u}, \mathbf{u}) = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$;
2. $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$;
3. $(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a(\mathbf{u}, \mathbf{w}) + b(\mathbf{v}, \mathbf{w})$.

$$\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})},$$

$$\mathrm{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

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We transfer the notion of perpendicular vectors from $I\!R^2$, $I\!R^3$ to V

In $I\!R^2$ holds

$$\mathbf{u} \perp \mathbf{v} \;\Leftrightarrow\; \text{Pythagoras formula holds,}$$
$$\Leftrightarrow\; \text{Diagonals of a parallelogram have the same length,}$$

Definition 1 *Let V, $(\ ,\ )$ be an inner product space, then $\mathbf{u}$, $\mathbf{v} \in V$ are called orthogonal or perpendicular $\Leftrightarrow$ $(\mathbf{u}, \mathbf{v}) = 0$.*

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Double-check, orthogonal vectors then form a generalized rectangle

Theorem 1 *Let V be a vector space and $\mathbf{u}$, $\mathbf{v} \in V$. Then,*

$$\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\| \quad \Leftrightarrow \quad (\mathbf{u}, \mathbf{v}) = 0.$$

Proof:

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u}, \mathbf{v}).$$
$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u}, \mathbf{v}).$$

then,

$$\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4(\mathbf{u}, \mathbf{v}).$$

□

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The vectors $\cos(x)$**,** $\sin(x)$ **which belong to** $C([0,2\pi])$ **are orthogonal**

$$\begin{aligned}(\cos(x),\sin(x)) &= \int_0^{2\pi} \sin(x)\cos(x)\,dx,\\ &= \frac{1}{2}\int_0^{2\pi} \sin(2x)\,dx,\\ &= -\frac{1}{4}\left(\cos(2x)|_0^{2\pi}\right),\\ &= 0.\end{aligned}$$

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Even subspaces can be orthogonal!

Definition 2 *Let* V*,* $(\ ,\ )$ *an inner product space and* $W \subset V$ *a subspace. Then* $W^{\perp}$ *is the orthogonal subspace, given by*

$$W^{\perp} = \{\mathbf{v} \in V \ : \ (\mathbf{v},\mathbf{u}) = 0, \quad \textit{for all } \mathbf{u} \in W\}.$$

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Orthogonal projection of a vector along any other vector is always possible

Fix $V, (\ ,\ )$, and $\mathbf{u} \in V$, with $\mathbf{u} \neq \mathbf{0}$.

Can any vector $\mathbf{x} \in V$ be decomposed in orthogonal parts with respect to $\mathbf{u}$?

That is, $\mathbf{x} = a\mathbf{u} + \mathbf{x}'$ with $(\mathbf{u}, \mathbf{x}') = 0$?

Is this decomposition unique?

Answer: Yes.

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Here is how to compute a and $\mathbf{x}'$

Theorem 2 *$V, (\ ,\ )$, an inner product vector space, and $\mathbf{u} \in V$, with $\mathbf{u} \neq \mathbf{0}$. Then, any vector $\mathbf{x} \in V$ can be uniquely decomposed as*

$$\mathbf{x} = a\mathbf{u} + \mathbf{x}', \quad \textit{where} \quad a = \frac{(\mathbf{x}, \mathbf{u})}{\|\mathbf{u}\|^2}.$$

Therefore,

$$\mathbf{x}' = \mathbf{x} - \frac{(\mathbf{x}, \mathbf{u})}{\|\mathbf{u}\|^2}\,\mathbf{u}, \quad \Rightarrow (\mathbf{u}, \mathbf{x}') = 0.$$

Orthogonal projection along a vector

Proof: Introduce $\mathbf{x}'$ by the equation $\mathbf{x} = a\mathbf{u} + \mathbf{x}'$. The condition $(\mathbf{u}, \mathbf{x}') = 0$ implies that

$$(\mathbf{x}, \mathbf{u}) = a(\mathbf{u}, \mathbf{u}), \quad \Rightarrow \quad a = \frac{(\mathbf{x}, \mathbf{u})}{\|\mathbf{u}\|^2},$$

then

$$\mathbf{x} = \frac{(\mathbf{x}, \mathbf{u})}{\|\mathbf{u}\|^2}\,\mathbf{u} + \mathbf{x}', \quad \Rightarrow \quad \mathbf{x}' = \frac{(\mathbf{x}, \mathbf{u})}{\|\mathbf{u}\|^2}\,\mathbf{u} - \hat{\mathbf{x}}.$$

This decomposition is unique, because, given a second decomposition $\mathbf{x} = b\mathbf{u} + \mathbf{y}'$ with $(\mathbf{u}, \mathbf{y}') = 0$, then

$$a\mathbf{u} + \mathbf{x}' = b\mathbf{u} + \mathbf{y}' \quad \Rightarrow \quad a = b,$$

from a multiplication by $\mathbf{u}$, and then,

$$\mathbf{x}' = \mathbf{y}'.$$

□

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Bases can be chose to be composed by mutually orthogonal vectors

Definition 3 *Let $V, (\ ,\ )$ be an n dimensional inner product space, and $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis of V.*

The basis is orthogonal $\Leftrightarrow$ $(\mathbf{u}_i, \mathbf{u}_j) = 0,$ *for all $i \neq j$.*

The basis is orthonormal $\Leftrightarrow$ *it is orthogonal, and* $\|\mathbf{u}_i\| = 1,$ *for all i,*

where $i,\ j = 1, \cdots, n$.

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To write x in an orthogonal basis is to decompose x along each basis vector direction

Theorem 3 *Let $V, (\ ,\ )$ be an n dimensional inner product vector space, and $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be an orthogonal basis. Then, any $\mathbf{x} \in V$ can be written as*

$$\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n,$$

with the coefficients have the form

$$c_i = \frac{(\mathbf{x}, \mathbf{u}_i)}{\|\mathbf{u}_i\|^2}, \quad i = 1, \cdots, n.$$

Proof: The set $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a basis, so there exist coefficients c_i such that $\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$. The basis is orthogonal, so multiplying the expression of $\mathbf{x}$ by $\mathbf{u}_i$, and recalling $(\mathbf{u}_i, \mathbf{u}_j) = 0$ for all $i \neq j$, one gets,

$$(\mathbf{x}, \mathbf{u}_i) = c_i(\mathbf{u}_i, \mathbf{u}_i).$$

The $\mathbf{u}_i$ are nonzero, so $(\mathbf{u}_i, \mathbf{u}_i) = \|\mathbf{u}_i\|^2 \neq 0$, so $c_i = (\mathbf{x}, \mathbf{u}_i)/\|\mathbf{u}_i\|^2$. □